Resolvent Estimates for the Laplacian in the Euclidean Space and on the Sphere

by
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Abstract

The thesis consists of two parts. In the first part, we prove an endpoint version of the uniform Sobolev inequalities in Kenig-Ruiz-Sogge [13]. Although strong inequality no longer holds for the pairs of exponents that are endpoints in the classical theorem of Kenig-Ruiz-Sogge [13], they enjoy restricted weak type inequality. The key ingredient in our proof is a method of interpolation first introduced by Bourgain [2]. Along with the proof of the endpoint uniform Sobolev inequalities, we give a complete description of the boundary case of Sogge's version of the Stein-Tomas restriction theorem in [18].

In the second part of the thesis, we turn to the sphere. More specifically, We extend the resolvent estimate on the sphere to exponent pairs off the line $\frac{1}{r} - \frac{1}{s} = \frac{2}{n}$. Since the condition $\frac{1}{r} - \frac{1}{s} = \frac{2}{n}$ on the exponent pairs is necessary for a uniform bound in the Euclidean case, one should not expect estimates off this line to be uniform for manifolds with constant curvature. The crucial step in our proof is an oscillatory integral theorem which easily leads to an (L^r, L^s) norm estimate on the operator H_k that projects onto the space of spherical harmonics of degree k. The rest of our proof then parallels that in Huang-Sogge [12].

READERS: Professor Christopher D. Sogge (Advisor) and Professor Joel Spruck

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Endpoint Version of Uniform Sobolev Inequalities in Euclidean Spaces

1.1 Introduction

In this thesis, we focus on the resolvent estimates for the Laplacian in Euclidean spaces or on manifolds. Let A be a closed operator on a (complex) Banach space X, with domain $D(A)$. The resolvent set of A, denoted $\rho(A)$, consists of complex numbers z such that the operator

$$A - zI \quad : \quad D(A) \to X$$

is one-to-one and onto. If $z \in \rho(A)$, we call the operator $R_z : X \to X$ defined by

$$R_z u \quad = \quad (A - zI)^{-1} u$$

a resolvent operator. Resolvent estimates for the Laplacian then, as the name suggests, is about estimates of the resolvent operators $(\Delta - zI)^{-1}$, where z belongs to appropriate regions of the complex plane. Put in a more straightforward way, we wish to compare the L^s norm of functions u and the L^r norm of functions $(\Delta + z)u$.

Among the first people to study resolvent estimates for the Laplacian systematically are C. E.

Kenig, A. Ruiz and C. D. Sogge. In 1986, they [13] showed that on $\mathbb{R}^n$ where $n \geqslant 3$, if the pair of exponents r, s satisfy the conditions

(a)

$$\frac{1}{r} - \frac{1}{s} = \frac{2}{n},$$

(b)

$$\min\left\{\left|\frac{1}{r} - \frac{1}{2}\right|, \left|\frac{1}{s} - \frac{1}{2}\right|\right\} > \frac{1}{2n},$$

then there exists a constant C, depending only on n, r and s, such that the following inequality holds:

$$\|u\|_{L^s(\mathbb{R}^n)} \leqslant C\|(\Delta + z)u\|_{L^r(\mathbb{R}^n)}, \quad u \in H^{2,r}(\mathbb{R}^n), \quad z \in \mathbb{C}. \tag{1.1.1}$$

More generally, we may replace the "$\Delta + z$" with any second order constant coefficient differential operator $P(D)$ whose principal part is "Δ":

$$\|u\|_{L^s(\mathbb{R}^n)} \leqslant C\|P(D)u\|_{L^r(\mathbb{R}^n)}, \quad u \in H^{2,r}(\mathbb{R}^n), \tag{1.1.2}$$

the constant C depending only on n, r and s. When $P(D)$ is the Laplacian Δ, (1.1.2) is just the classical Sobolev inequality. However, the striking thing about inequalities (1.1.1) and (1.1.2) is the independence of the constant C from the complex number z or the coefficients of the zero and first order terms in $P(D)$. Hence, these inequalities are referred to as uniform Sobolev inequalities by Kenig, Ruiz and Sogge.

These uniform Sobolev inequalities have important applications in partial differential equations, one of which is the unique continuation results they lead to. For instance, Kenig, Ruiz and Sogge proved in that same paper [13], along with several other strong and useful unique continuation theorems, that assuming $p = \frac{2n}{n+2}$, $P(D)$ a differential operator as above and $V(x)$ a function in $L^{\frac{n}{2}}(\mathbb{R}^n)$, if $u \in H^{2,p}(\mathbb{R}^n)$ is supported in one side of a hyperplane of $\mathbb{R}^n$ and satisfies $|P(D)u(x)| \leqslant |V(x)u(x)|$, then u is identically 0 on all of $\mathbb{R}^n$. Details are in the second part of [13].

In addition to the merits just mentioned, Kenig, Ruiz and Sogge's theorems on uniform Sobolev inequalities are sharp in terms of the exponents-they proved that conditions (a) and (b) above on the exponents are necessary so that (1.1.1) and (1.1.2) hold with the constant C independent of the complex number z or the coefficients of the zero and first order terms of $P(D)$. Visualized in the plane whose two axes are $\frac{1}{r}$ and $\frac{1}{s}$, these permissible exponent pairs constitute the *open* line segment connecting the point $\alpha(\frac{n+1}{2n}, \frac{n-3}{2n})$ and its dual point $\beta(\frac{n+3}{2n}, \frac{n-1}{2n})$. See the picture below.

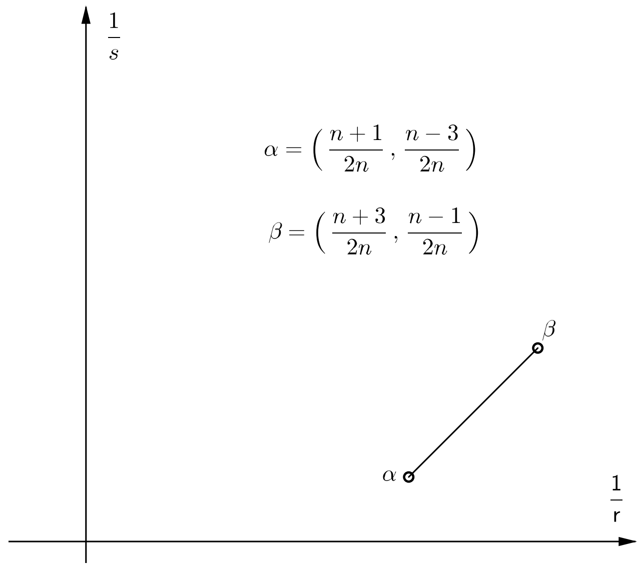

Figure 1.1: The open line segment for uniform Sobolev inequalities

We however, tackle the endpoint case of Kenig, Ruiz and Sogge's classical theorems. In a joint work [16] with Y. Xi and C. Zhang, we showed that although strong inequality does not hold at the endpoints α and β, they enjoy restricted weak type inequality. Specifically,

Theorem 1. *Suppose $n \geqslant 3$. If $r = \frac{2n}{n+1}, s = \frac{2n}{n-3}$, or $r = \frac{2n}{n+3}, s = \frac{2n}{n-1}$ are one pair of the endpoints, then the inequality*

$$\|u\|_{L^{s,\infty}(\mathbb{R}^n)} \leqslant C\|(\Delta + z)u\|_{L^{r,1}(\mathbb{R}^n)}, \quad u \in \mathscr{S}(\mathbb{R}^n), \quad z \in \mathbb{C} \tag{1.1.3}$$

holds, with the constant C depending only on n.

Here, the spaces $L^{s,\infty}(\mathbb{R}^n)$ and $L^{r,1}(\mathbb{R}^n)$ denote Lorentz spaces. Their definitions and properties, together with what it means by restricted weak type inequality, will be given in the preliminaries on Lorentz spaces and interpolation (Section 1.2). We also remark that S. Gutierrez [11] obtained restricted weak type resolvent estimates as in Theorem 1 at points $(\frac{n+1}{2n}, \frac{(n-1)^2}{2n(n+1)})$ and $(\frac{n^2+4n-1}{2n(n+1)}, \frac{n-1}{2n})$. They are A and B in Figure 1.2. The estimates cannot be uniform, but depend on z, because the exponent pairs are not on the line $\frac{1}{p} - \frac{1}{q} = \frac{2}{n}$.

Furthermore, we too are able to add first order terms into our theorem when $n > 3$, and obtain the general version below. The exclusion of the three dimensional case for the general version is worthy of noticing; we will state the reason after the proof of this theorem, when it becomes clear.

Theorem 2. *Let $n > 3$. If the pair of exponents (r, s) are as in Theorem 1, then there exists a constant C, depending only on n, such that whenever $P(D)$ is a second order constant coefficient differential operator whose principal part is "Δ", one has the restricted weak type inequality*

$$\|u\|_{L^{s,\infty}(\mathbb{R}^n)} \leqslant C\|P(D)u\|_{L^{r,1}(\mathbb{R}^n)}, \quad u \in \mathscr{S}(\mathbb{R}^n). \tag{1.1.4}$$

By the real interpolation that will be stated in the next section, the above two theorems imply the corresponding results in Kenig-Ruiz-Sogge [13]. The key ingredient in proving Theorem 1 is an interpolation technique first introduced by J. Bourgain [2] when he was proving a bound for the spherical maximal function, and we first noticed this technique in a paper [1] of J.-G. Bak and A. Seeger. To apply the interpolation, we need a variant of Stein's oscillatory integral theorem due to Sogge [18]. Inspired by the proof of Theorem 1, we go on to completely settle the boundary case of Sogge's version of Stein-Tomas restriction theorem for the Fourier transform in [18]-the operators in these two problems have similar kernels. This result will play an essential role in the proof of the general case Theorem 2. Once we have done these work, we will need the adaptations of a few classical results for Lebesgue spaces to Lorentz spaces. Kenig-Ruiz-Sogge's general version follows from their special version by a localization argument [13, P.335-337]. With the above preparations, we too are able to carry out that localization argument to obtain Theorem 2.

1.2 Preliminaries: Lorentz Spaces and Interpolation

Two most common types of inequalities for operators in harmonic analysis are strong type and weak type. Given exponents $1 \leqslant p, q \leqslant \infty$, an operator T between function spaces is said to be of strong type (p, q) if $\|Tf\|_{L^q} \leqslant \|f\|_{L^p}$. For $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$, T is called weak type (p, q) if

$$\mu(\{x : |f(x)| > \alpha\}) \leqslant C\Big(\frac{\|f\|_{L^p}}{\alpha}\Big)^q, \quad \forall \alpha > 0;$$

and when $q = \infty$, the definition of weak type (p, q) is the same as that of strong type, according to convention. Here, $\mu(\cdot)$ means Lebesgue measure. It is customary to denote

$$\lambda(\alpha) = \mu(\{x : |f(x)| > \alpha\})$$

and $\lambda(\alpha)$ is called the distribution function associated with f.

A famous theorem of Marcinkiewicz on interpolation says that if T is a subadditive operator, meaning $|T(f_1+f_2)(x)| \leqslant |T(f_1)(x)|+|T(f_2)(x)|$, $\forall x$, and is of weak type (p_i,q_i), where $1\leqslant p_i,q_i\leqslant \infty$, $i=1,2$ and $q_1\neq q_2$, then T is of strong type (p_θ,q_θ), where

$$\frac{1}{p_\theta}=\frac{\theta}{p_1}+\frac{1-\theta}{p_2},\quad \frac{1}{q_\theta}=\frac{\theta}{q_1}+\frac{1-\theta}{q_2},$$

$$0<\theta<1.$$

Lorentz spaces, also known as $L^{p,q}$ spaces, were introduced so as to further weaken the hypotheses under which Marcinkiewicz interpolation theorem holds.

Fix $1\leqslant p,q\leqslant\infty$. For a measurable function f, if $\lambda(s)$ denotes its distribution function, we define the nonincreasing rearrangement $f^*(t)$, $t>0$ of f by letting

$$f^*(t)=\inf\{s:\lambda(s)\leqslant t\}.$$

When $1\leqslant p<\infty$, $1\leqslant q<\infty$, the Lorentz space $L^{p,q}$ consists of all those measurable functions satisfying

$$\|f\|^*_{L^{p,q}}=\Big(\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}}f^*(t)]^q\frac{dt}{t}\Big)^{\frac{1}{q}}<\infty, \tag{1.2.1}$$

and when $1\leqslant p\leqslant\infty$, $q=\infty$, the defining formula naturally becomes

$$\|f\|^*_{L^{p,q}}=\sup_{t>0} t^{\frac{1}{p}}f^*(t)<\infty. \tag{1.2.2}$$

. In words, omitting for the moment the constant $\frac{q}{p}$ whose appearance will be explained later, $\|f\|^*_{L^{p,q}}$ equals the L^q norm of the function $t^{\frac{1}{p}}f^*(t)$ on $\mathbb{R}$ with respect to the measure $\frac{dt}{t}$. Note that when $p=\infty$, we only define $L^{p,q}$ for $q=\infty$.

It is not hard to prove that f and f^* have the same distribution function. Therefore, another formula for $\|f\|^*_{L^{p,q}}$, $p\neq\infty$, which is also widely used, is the following:

$$\|f\|^*_{L^{p,q}}=\Big(q\int_0^\infty [t\lambda(t)^{\frac{1}{p}}]^q\frac{dt}{t}\Big)^{\frac{1}{q}} \tag{1.2.3}$$

for $1\leqslant p<\infty$, $1\leqslant q<\infty$;

$$\|f\|^*_{L^{p,q}}=\sup_{t>0} t\lambda(t)^{\frac{1}{p}} \tag{1.2.4}$$

for $1\leqslant p<\infty$, $q=\infty$.

One immediately notices that if $f=\mathbb{1}_E$ is the characteristic function of a measurable set of finite measure, we have

$$\|\mathbb{1}_E\|^*_{L^{p,q}} = \mu(E)^{\frac{1}{p}},$$

that is, $\|f\|^*_{L^{p,q}}$ agrees with the L^p norm regardless of q whenever f is a characteristic function. Moreover, verifying first for simple functions and then passing to limit, we deduce that

$$\|f\|^*_{L^{p,p}} = \|f\|_{L^p}, \quad 1 \leqslant p \leqslant \infty.$$

Therefore, the $L^{p,p}$ spaces are just L^p spaces. These are the reasons for introducing the constant $\frac{q}{p}$ in the definition of $L^{p,q}$ spaces. Finally regarding the definition, we have the order relation

$$\|f\|^*_{L^{p,q_2}} \leqslant \|f\|^*_{L^{p,q_1}} \tag{1.2.5}$$

whenever $q_1 \leqslant q_2$, hence the inclusion relation $L^{p,q_1} \subseteq L^{p,q_2}$. The above few observations indicate that Lorentz spaces are an extension of Lebesgue spaces, with $L^{p,1}$ the smallest such extension and $L^{p,\infty}$ the largest.

Despite all good traits about the definition of $L^{p,q}$ spaces, the disappointing truth is $\|\cdot\|^*_{L^{p,q}}$ so defined are usually not norms, as Minkowski's inequality does not hold for most of them. This is why we always avoided saying "the norm $\|\cdot\|^*_{L^{p,q}}$" in seemingly appropriate situations; it is only a quasi-norm. One overcomes this obstacle by introducing a closely connected equivalent definition that indeed gives norms. The idea is that we replace $f^*(t)$ in the definition with its average over the interval $(0,t)$. Specifically, for $t > 0$, let

$$m(t) = \frac{1}{t}\int_0^t f^*(u)du,$$

and define

$$\|f\|_{L^{p,q}} = \Big(\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}}m(t)]^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$

when $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, and

$$\|f\|_{L^{p,q}} = \sup_{t>0} t^{\frac{1}{p}}m(t)$$

when $1 \leqslant p \leqslant \infty$, $q = \infty$. If $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $\|\cdot\|_{L^{p,q}}$ is equivalent to $\|\cdot\|^*_{L^{p,q}}$:

$$\|f\|^*_{L^{p,q}} \leqslant \|f\|_{L^{p,q}} \leqslant \frac{p}{p-1}\|f\|^*_{L^{p,q}}.$$

In the case of $p = 1$ however, this formula is unsatisfactory. Simple limiting argument shows that $\|f\|_{L^{1,\infty}} = \|f\|_{L^1}$, while $\|f\|_{L^{1,q}} < \infty$ only if $f \equiv 0$ for $1 \leqslant q < \infty$. So the formula for $\|\cdot\|_{L^{1,q}}$ does not provide a desirable extension of the L^1 space as does $\|\cdot\|^*_{L^{1,q}}$.

The relevant fact about the averaging function $m(t)$ is

$$m(t) = \sup_{\mu(E) \leqslant t} \int_E |f| dx,$$

hence $\|\cdot\|_{L^{p,q}}$ satisfies Minkowski's inequality. In fact, when $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $\|\cdot\|_{L^{p,q}}$ is a norm that makes $L^{p,q}$ a Banach space. This is the main advantage of using $\|\cdot\|_{L^{p,q}}$ over $\|\cdot\|^*_{L^{p,q}}$. However, realizing that the initial definition is easy to manipulate while the equivalent one is virtually impossible to handle, the significance of the former one is not to be neglected nor regretted. In a word, each has its advantages and drawbacks, and we will shuttle between them in a part of the subsequent development.

Now we are in a good position to weaken the conditions in Marcinkiewicz interpolation theorem. By the definition of $\|\cdot\|^*_{L^{p,q}}$, an operator T is of weak type (p, q) if

$$\|Tf\|^*_{L^{q,\infty}} \leqslant C\|f\|_{L^p}.$$

Because of the order relation

$$\|f\|_{L^p} \leqslant \|f\|^*_{L^{p,1}}, \quad p \geqslant 1,$$

it is natural to consider the inequality

$$\|Tf\|^*_{L^{q,\infty}} \leqslant C\|f\|^*_{L^{p,1}}. \tag{1.2.6}$$

T is said to enjoy restricted weak type (p, q) inequality if it satisfies (1.2.6).

Another way of observing the pertinence of restricted weak type inequality to weak type inequality is that restricted weak type inequality amounts to applying weak type inequality to characteristic functions only. More precisely, if

$$\|T\mathbb{1}_E\|^*_{L^{q,\infty}} \leqslant C\mu(E)^{\frac{1}{p}}$$

holds for any measurable set E of finite measure, then

$$\|Tf\|^*_{L^{q,\infty}} \leqslant C\|f\|^*_{L^{p,1}}$$

holds for all functions f. (The two constants are probably different though.) Thus, to prove restricted weak type inequality, it suffices to consider characteristic functions.

The crucial interpolation theorem that we will use is the following. It is referred to as the real interpolation in some literature, to distinguish it from the complex interpolation.

Theorem 3. *Suppose T is a subadditive operator of restricted weak type (p_i, q_i), $i = 1, 2$, with $p_1 \neq p_2$ and $q_1 \neq q_2$, then there exists a constant $C = C(\theta)$ such that*

$$\|Tf\|^*_{L^{q,s}} \leqslant C\|f\|^*_{L^{p,s}}, \tag{1.2.7}$$

where $1 \leqslant s \leqslant \infty$,

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad 0 < \theta < 1.$$

If $q_1 \geqslant p_1$ and $q_2 \geqslant p_2$, then it follows that p, q as in the theorem always satisfy $q \geqslant p$, and consequently, by the order relation of $L^{p,q}$ norms, restricted weak type inequalities at the endpoints (p_1, q_1) and (p_2, q_2) imply strong inequalities for all intermediate exponent pairs (p, q):

$$\|Tf\|_{L^q} \leqslant C\|f\|_{L^p}.$$

This is where the significance of the interpolation theorem lies. Estimates for exponent pairs on an open line segment might reduce to restricted weak type inequalities at the two endpoints, which in turn are just weak type inequalities for characteristic functions. As a famous example, in both Bourgain's and Wolff's proofs of the Kakeya maximal function inequality, the question first reduces to restricted weak type inequality at one end. And then the essence becomes combinatoric arguments due to the nature of the Kakeya problem.

Emphatically, the hypotheses $p_1 \neq p_2$, $q_1 \neq q_2$ cannot be discarded. This means, on the plane whose two axes are $\frac{1}{p}$ and $\frac{1}{q}$, we cannot interpolate between restricted weak type inequalities at two points on a horizontal or vertical line.

Lorentz spaces share many properties with Lebesgue spaces, although they reject many other. One of those common properties is duality characterization. Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. Obeying the convention, we use p' and q' to signify conjugates of p and q, meaning $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then if $f \in L^{p,q}$, its $L^{p,q}$ norm can be characterized as

$$c \sup\{\Big|\int_{\mathbb{R}^n} f\bar{g}dx\Big| : \|g\|_{L^{p',q'}} \leqslant 1\} \leqslant \|f\|_{L^{p,q}} \leqslant C \sup\{\Big|\int_{\mathbb{R}^n} f\bar{g}dx\Big| : \|g\|_{L^{p',q'}} \leqslant 1\}. \tag{1.2.8}$$

The condition $f \in L^{p,q}$ may be dropped if f is nonnegative. Such result breaks down at $p = 1$ and $p = \infty$.

In addition, when $1 < p < \infty$, $1 \leqslant q < \infty$, the space of simple functions is dense in $L^{p,q}$. However, when $q = \infty$, this statement is not true. For instance, one cannot approximate $|x|^{-\frac{n}{p}} \in L^{p,\infty}$ by simple functions.

Another proposition for Lebesgue spaces that passes over to Lorentz spaces is Holder's inequality. The result is due to R. O'Neil [14].

Proposition 1. *Suppose* $1 \leqslant p_1, p_2, p < \infty$, $1 \leqslant q_1, q_2, q \leqslant \infty$ *satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$\|fg\|_{L^{p,q}} \leqslant C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}},$$

if the expression on the right is finite.

1.3 Bourgain's Interpolation

Recall that we aim to prove restricted weak type inequalities at the endpoints in Kenig, Ruiz and Sogge's classical theorem. For this we need, as pointed out in the introduction, a type of interpolation credited to Bourgain that gives rise to restricted weak type inequalities. Besides Bak-Seeger [1], there is an elaboration on the abstract theory, developed for fairly general normed vector spaces, in Carbery-Seeger-Wainger-Wright [4]. We are not going into such abstractness and generality, but would rather accommodate the result to our specific setting, that of L^p spaces. The interpolation goes

Lemma 1. *If an operator T between function spaces is the sum of the operators T_j: $T = \sum_{j=1}^{\infty} T_j$, and if each T_j satisfies the estimates*

$$\|T_j\|_{L^{p_1} \to L^{q_1}} \leqslant M_1 2^{\beta_1 j}, \tag{1.3.1}$$

$$\|T_j\|_{L^{p_2} \to L^{q_2}} \leqslant M_2 2^{-\beta_2 j}, \tag{1.3.2}$$

for some constants $M_1 > 0$, $M_2 > 0$ and $\beta_1 > 0$, $\beta_2 > 0$, then T, the sum of the T_j, enjoys restricted *weak type inequality between the intermediate spaces:*

$$\|T\|_{L^{p,1}\to L^{q,\infty}} \leqslant CM_1^{\theta}M_2^{1-\theta}, \quad \textit{where} \tag{1.3.3}$$

$$\theta = \frac{\beta_2}{\beta_1+\beta_2},$$
$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2},$$

and C depends only on β_1 and β_2.

R. Frank and L. Schimmer's idea in their proof of a bound for the Hadamard parametrix in a very recent paper [9] actually directs us to a proof of Bourgain's interpolation theorem in the setting of Lebesgue spaces. We record their idea here.

Proof of Lemma 1 By the property of Lorentz spaces in Section 1.2, it suffices to prove (1.3.3) for characteristic functions, i.e. we need only show the inequality

$$\Big(\sup_{\lambda>0}\lambda^q\mu\big(\{x:|T\mathbb{1}_E|>\lambda\}\big)\Big)^{\frac{1}{q}} \leqslant CM_1^{\theta}M_2^{1-\theta}\mu(E)^{\frac{1}{p}}, \tag{1.3.4}$$

where E is a measurable set of finite measure, and $\mu(\cdot)$ indicates the measure of a set. For convenience, we denote

$$A = \{x:|T\mathbb{1}_E|>\lambda\}.$$

In what follows, it is necessary to ensure that A has finite measure, but this is in fact a simple consequence of the second estimate (1.3.2) for T_j in the Lemma. To see why, first note that by summing the T_j, we conclude that T is a bounded operator from L^{p_2} to L^{q_2}. This fact together with Tchebyshev's inequality then immediately yields

$$\lambda^{q_2}\mu(A) \leqslant \int \big(|T\mathbb{1}_E|\big)^{q_2}\mathrm{d}\mu \leqslant \mu(E)^{\frac{q_2}{p_2}} < \infty.$$

Still by Tchebyshev's inequality, we have

$$\mu(A) \leqslant \frac{1}{\lambda}\int_A |T\mathbb{1}_E|\mathrm{d}\mu. \tag{1.3.5}$$

The trick now is to split the sum $T = \sum_{j=1}^{\infty} T_j$ into two parts, a finite one and the rest infinite one:

$$T^{(1)} = \sum_{j=1}^{\rho} T_j, \quad T^{(2)} = \sum_{j=\rho+1}^{\infty} T_j,$$

where ρ is to be specified later. Then by Holder's inequality,

$$\begin{aligned} \mu(A) &\leqslant \frac{1}{\lambda}\Big(\int_A |T^{(1)}\mathbb{1}_E|\mathrm{d}\mu + \int_A |T^{(2)}\mathbb{1}_E|\mathrm{d}\mu\Big) \\ &\leqslant \frac{1}{\lambda}\Big(\|T^{(1)}\mathbb{1}_E\|_{L^{q_1}}\mu(A)^{\frac{1}{q_1'}} + \|T^{(2)}\mathbb{1}_E\|_{L^{q_2}}\mu(A)^{\frac{1}{q_2'}}\Big), \end{aligned} \tag{1.3.6}$$

where as before, q_1' and q_2' denote conjugates of q_1 and q_2 respectively.

For the finite sum $T^{(1)}$, we apply the first estimate (1.3.1), the positive exponential bound, in the lemma, and for the remaining infinite $T^{(2)}$, we apply (1.3.2), the negative exponential bound. These give

$$\begin{aligned} \|T^{(1)}\mathbb{1}_E\|_{L^{q_1}} &\leqslant \sum_{j=1}^{\rho} M_1 2^{\beta_1 j}\|\mathbb{1}_E\|_{L^{p_1}} = \sum_{j=1}^{\rho} M_1 2^{\beta_1 j}\mu(E)^{\frac{1}{p_1}} \\ &\leqslant C M_1 2^{\beta_1 \rho}\mu(E)^{\frac{1}{p_1}}, \end{aligned} \tag{1.3.7}$$

and

$$\begin{aligned} \|T^{(2)}\mathbb{1}_E\|_{L^{q_2}} &\leqslant \sum_{j=\rho+1}^{\infty} M_2 2^{-\beta_2 j}\|\mathbb{1}_E\|_{L^{p_2}} = \sum_{j=\rho+1}^{\infty} M_2 2^{-\beta_2 j}\mu(E)^{\frac{1}{p_2}} \\ &\leqslant C M_2 2^{-\beta_2 \rho}\mu(E)^{\frac{1}{p_2}}. \end{aligned} \tag{1.3.8}$$

Substituting the above two inequalities in (1.3.6), we have

$$\mu(A) \leqslant \frac{C}{\lambda}\Big(M_1 2^{\beta_1\rho}\mu(E)^{\frac{1}{p_1}}\mu(A)^{\frac{1}{q_1'}} + M_2 2^{-\beta_2\rho}\mu(E)^{\frac{1}{p_2}}\mu(A)^{\frac{1}{q_2'}}\Big). \tag{1.3.9}$$

If we minimized the right-side of the last inequality by choosing appropriate ρ, then we would obtain

$$\mu(A) \leqslant \frac{C}{\lambda} M_1^{\theta} M_2^{1-\theta}\mu(E)^{\frac{1}{p}}\mu(A)^{1-\frac{1}{q}}, \tag{1.3.10}$$

which would imply the estimate (1.3.4) that we set out to prove. The ρ achieving the favorable result is such that

$$2^{\rho} = \Big(\frac{M_1}{M_2}\mu(E)^{\frac{1}{p_2}-\frac{1}{p_1}}\mu(A)^{\frac{1}{q_2'}-\frac{1}{q_1'}}\Big)^{\frac{1}{\beta_1+\beta_2}}. \tag{1.3.11}$$

However, the ρ here must be a nonnegative integer, so we need a few more lines to guarantee that the above reasoning goes through well. If

$$\frac{M_1}{M_2}\mu(E)^{\frac{1}{p_2}-\frac{1}{p_1}}\mu(A)^{\frac{1}{q_2'}-\frac{1}{q_1'}} > 1,$$

then we may let ρ be the nonnegative integer such that

$$2^\rho < \left(\frac{M_1}{M_2}\mu(E)^{\frac{1}{p_2}-\frac{1}{p_1}}\mu(A)^{\frac{1}{q_2'}-\frac{1}{q_1'}}\right)^{\frac{1}{\beta_1+\beta_2}} \leqslant 2^{\rho+1}.$$

This choice of ρ will lead us to the same estimate as (1.3.10), with a worse C. If however

$$\frac{M_1}{M_2}\mu(E)^{\frac{1}{p_2}-\frac{1}{p_1}}\mu(A)^{\frac{1}{q_2'}-\frac{1}{q_1'}} \leqslant 1,$$

then we may simply let ρ be 0 and sum the negative exponential bounds. The result is

$$\mu(A) \leqslant \frac{1}{\lambda} M_2 \mu(E)^{\frac{1}{p_2}}\mu(A)^{\frac{1}{q_2'}}. \tag{1.3.12}$$

Rearranging terms, this is equivalent to

$$\lambda\mu(A)^{\frac{1}{q}} \leqslant M_1^\theta M_2^{1-\theta}\mu(E)^{\frac{1}{p}} \cdot \left(\frac{M_1}{M_2}\mu(E)^{\frac{1}{p_2}-\frac{1}{p_1}}\mu(A)^{\frac{1}{q_2'}-\frac{1}{q_1'}}\right)^{\frac{\beta_2}{\beta_1+\beta_2}}.$$

Considering our assumption, the above is in fact a stronger result than we need. We thence concludes the proof of Bourgain's interpolation. □

1.4 Proof of Theorem 1

We are ready to prove Theorem 1. First of all, we need some reductions. It suffices to prove the theorem for one endpoint, say $(\frac{n+1}{2n}, \frac{n-3}{2n})$, because the other follows from duality. Furthermore, noting the gap condition $\frac{1}{r} - \frac{1}{s} = \frac{2}{n}$ on the exponents, we are able to reduce the theorem to the case where z has unit length, $|z| = 1$, after a simple rescaling argument. Finally, by continuity, the detail of which we will give at the end of the proof, we may assume that $\mathrm{Im} z \neq 0$.

This last reduction enables us to study $(-|\xi|^2 + z)^{-1}$, whose inverse Fourier transform is the fundamental solution of the operator "$\Delta + z$" in our theorem. Therefore, our theorem is a consequence of the following estimate for a multiplier operator:

$$\Big\|\Big\{\frac{\hat{u}(\xi)}{-|\xi|^2+z}\Big\}^{\vee}\Big\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)} \leqslant C\|u\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}. \tag{1.4.1}$$

This in turn, amounts to the inequality for a convolution operator

$$\Big\|u(x) * \Big\{\frac{1}{-|\xi|^2+z}\Big\}^{\vee}(x)\Big\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)} \leqslant C\|u\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}. \tag{1.4.2}$$

The proof of the theorem then transforms to the study of the kernel of this convolution operator: $K(x) = \Big\{\frac{1}{-|\xi|^2+z}\Big\}^{\vee}(x)$. The expression for this kernel is actually already in literature, e.g. Gelfand-Shilov [10], and by their work,

$$K(x) = \Big(\frac{z}{|x|^2}\Big)^{\frac{n-2}{4}} K_{\frac{n-2}{2}}(\sqrt{z|x|^2}), \tag{1.4.3}$$

where

$$K_\nu(w) = \int_0^\infty e^{-w\mathrm{cosht}}\mathrm{cosh}(\nu t)\mathrm{d}t \tag{1.4.4}$$

denotes the modified Bessel function. Along with the expression for $K(x)$, we will also need the following facts about the Bessel function, all of which are contained in [13, P. 339].

First, a change of variable $u = e^t$ in the expression (1.4.4) for $K_\nu(w)$ immediately yields

$$|K_\nu(w)| \leqslant C|w|^{-|\mathrm{Re}(\nu)|}, \tag{1.4.5}$$

for $|w| \leqslant 1$ and $\mathrm{Re}(w) > 0$, where the constant C depends only on ν. Second, applying the formula (see [6, P. 19])

$$\Gamma(\nu+\frac{1}{2})K_\nu(w) = \Big(\frac{\pi}{2w}\Big)^{\frac{1}{2}} e^{-w}\int_0^\infty e^{-t}t^{\nu-\frac{1}{2}}\Big(1+\frac{t}{2w}\Big)^{\nu-\frac{1}{2}}\mathrm{d}t, \tag{1.4.6}$$

which is valid when $\mathrm{Re}\nu \geqslant 0$, we obtain the behavior of the Bessel function for large $|w|$:

$$|K_\nu(w)| \leqslant Ce^{-\mathrm{Re}(w)}|w|^{-\frac{1}{2}} \tag{1.4.7}$$

whenever $|w| \geqslant 1$ and $\mathrm{Re}(w) > 0$. Finally, formula (1.4.6) in fact tells us that

$$K_\nu(w) = a_\nu(w)w^{-\frac{1}{2}}e^{-w}, \tag{1.4.8}$$

for $\mathrm{Re}(w) > 0$, where the function $a_\nu(w)$ enjoys the decaying property

$$\left|\left(\frac{\partial}{\partial w}\right)^\alpha a_\nu(w)\right| \leqslant C_\alpha |w|^{-\alpha}. \tag{1.4.9}$$

With these preparations, we embark on the task of proving the estimate for the convolution operator $K(x)$. The idea is to treat the part of $K(x)$ inside the unit sphere and the part outside separately, hence we break $K(x) = K_1(x) + K_2(x)$, where $K_1(x)$ is defined to be identical with $K(x)$ in the unit sphere $|x| \leqslant 1$, and equal 0 elsewhere. By estimate (1.4.5), considering the expression for $K(x)$, we easily obtain that

$$|K_1(x)| \leqslant C|x|^{-(n-2)}. \tag{1.4.10}$$

Then the desired, indeed strong, inequality

$$\|u(x) * K_1(x)\|_{L^{\frac{2n}{n-3}}(\mathbb{R}^n)} \leqslant C\|u\|_{L^{\frac{2n}{n+1}}(\mathbb{R}^n)} \tag{1.4.11}$$

follows from Hardy-Littlewood-Sobolev inequality whenever the dimension $n > 3$, noticing that the exponents $(\frac{n+1}{2n}, \frac{n-3}{2n})$ are on the line $\frac{1}{r} - \frac{1}{s} = \frac{2}{n}$. If $n = 3$ however, we cannot apply Hardy-Littlewood Sobolev inequality, because one exponent $\frac{2n}{n-3}$ is ∞ then. Nevertheless, the restricted weak type estimate

$$\|u(x) * K_1(x)\|_{L^\infty(\mathbb{R}^3)} \leqslant \|u\|_{L^{\frac{3}{2},1}(\mathbb{R}^3)} \|K_1\|_{L^{3,\infty}(\mathbb{R}^3)} \leqslant C\|u\|_{L^{\frac{3}{2},1}(\mathbb{R}^3)} \tag{1.4.12}$$

still holds, by the Holder's inequality for Lorentz spaces, Proposition 1 in Section 1.2.

After that, we turn to our analysis of $K_2(x)$, the part of $K(x)$ away from the origin. Applying (1.4.7) yields the estimate

$$|K_2(x)| \leqslant C|x|^{-\frac{n-1}{2}} e^{-|x|\cos(\frac{1}{2}\arg z)}. \tag{1.4.13}$$

Because of the exponential term, which may have the decaying property we desire, we separate the case where $\arg z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, from the case where $\arg z \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$. For the former situation, as just mentioned, the effect of the exponential decay yields the strong estimate

$$\|u(x) * K_2(x)\|_{L^{\frac{2n}{n-3}}(\mathbb{R}^n)} \leqslant C\|u\|_{L^{\frac{2n}{n+1}}(\mathbb{R}^n)}, \tag{1.4.14}$$

which is a result from Young's inequality.

The difficult situation is the latter one, and this is where Bourgain's interpolation comes into play. By the expression (1.4.8) for the Bessel function,

$$K_2(x) = |x|^{-\frac{n-1}{2}} e^{-i|x|\sin(\frac{1}{2}\arg z)} e^{i\frac{n-3}{4}\arg z} e^{-|x|\cos\frac{1}{2}\arg z} a_{\frac{n-2}{2}}(|x|e^{i\frac{1}{2}\arg z}), \tag{1.4.15}$$

where $a_{\frac{n-2}{2}}(w)$ satisfies the decaying property (1.4.9). To apply Bourgain's interpolation, we dyadically decompose $K_2(x)$. Fix a smooth function $\eta(x)$ that has support in $\{x : |x| < 1\}$ and is equal to 1 for $|x| < \frac{1}{2}$. Denote $\delta(x) = \eta(x) - \eta(2x)$. Then let $\beta_0(x) = \eta(x)$, and for each $j > 1$, let $\beta_j(x) = \delta(2^{-j}x)$. It is easy to verify that $\sum_{j=0}^{\infty} \beta_j(x) = 1$.

For each $j \geqslant 0$, consider the operator T_j given by the kernel $K_{2,j}(x) = \beta_j(x)K_2(x)$, i.e. $T_j u = u * K_{2,j}$. To tackle kernels with appearance like that of $K_{2,j}(x)$, we need invoke the following variant of Stein's oscillatory integral theorem.

Lemma 2. *Let $n \geqslant 3$. Suppose that $1 \leqslant p \leqslant 2$, $q = \frac{n+1}{n-1}p'$; in other words, the pair of exponents (p, q) lies on the closed line segment joining $E(\frac{1}{2}, \frac{n-1}{2(n+1)})$ and $(1, 0)$. Then, given a kernel of the form*

$$L(x) = \delta(x)b(x)e^{i\lambda|x|}|x|^{-\frac{n-1}{2}}, \tag{1.4.16}$$

where $\lambda \neq 0$, $\delta(x)$ is a smooth function supported in $\{x \in \mathbb{R}^n : \frac{1}{4} \leqslant |x| \leqslant 1\}$, $b(x) \in C^\infty(\mathbb{R}^n)$, and $|((\frac{\partial}{\partial x})^\alpha b)(x)| \leqslant C_\alpha |x|^{-|\alpha|}$, we have the inequality

$$\|L * f\|_{L^q(\mathbb{R}^n)} \leqslant C|\lambda|^{-\frac{n}{q}} \|f\|_{L^p(\mathbb{R}^n)}, \tag{1.4.17}$$

where the constant C depends only on the function $\delta(x)$ and finitely many of the C_α above.

This oscillatory integral theorem follows from Stein's oscillatory integral theorem [22]. See also [13, P. 341] and [19, P. 63]. It holds for pairs of exponents lying on the closed line segment connecting $(\frac{1}{p}, \frac{1}{q}) = (1, 0)$ and $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{2}, \frac{n-1}{2(n+1)})$. Because of this, we are tempted to interpolate between the intersection of the line segment $\frac{1}{r} - \frac{1}{s} = \frac{2}{n}$ for resolvent estimate with the line segment on which the oscillatory integral theorem holds, and the point $(\frac{2}{n}, 0)$ on the $\frac{1}{r}$ axis. These two points that we want to interpolate between are shown in Figure 1.2, labeled P and Q. It is easy to calculate that P has coordinates $(\frac{n^2+n+2}{2n^2}, \frac{n^2-3n+2}{2n^2})$.

There is, however, a small trouble with the three dimensional case. When $n = 3$, Q actually coincides with our target pair of exponents $(\frac{n+1}{2n}, \frac{n-3}{2n})$, hence interpolation cannot produce the desired result. Fortunately, this is not too much of a trouble, and we are able to remedy it by interpolating between two other points. In the following, we treat the case $n > 3$ first, postponing

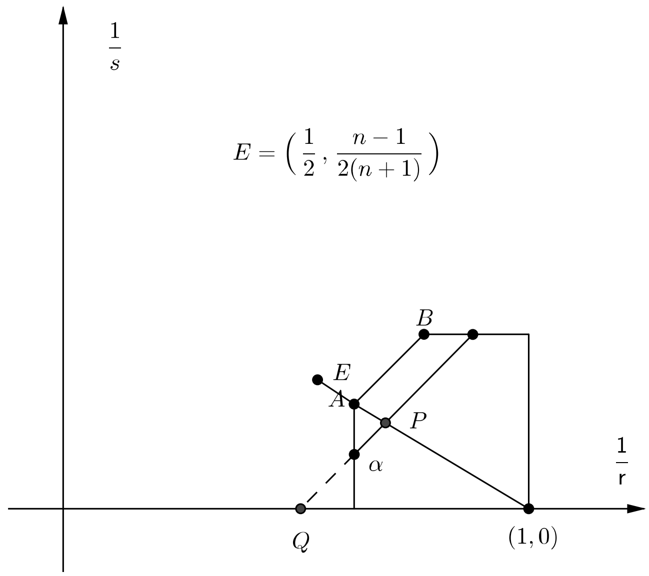

Figure 1.2: The interpolation diagram for the resolvent estimates

the exceptional three dimensional case until later.

Here is the interpolation process. At Q, since $K_{2,j}(x) \leqslant C|x|^{-\frac{n-1}{2}}$ by (1.4.7) and $K_{2,j}(x)$ is supported in $\{x : 2^{j-2} \leqslant |x| \leqslant 2^j\}$, Young's inequality yields

$$\begin{aligned} \|K_{2,j}(x) * u(x)\|_{L^\infty(\mathbb{R}^n)} &\leqslant \|u\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \|K_{2,j}\|_{L^{\frac{n}{n-2}}(\mathbb{R}^n)} \\ &\leqslant C 2^{-j\frac{n-1}{2}} (2^{jn})^{\frac{n-2}{n}} \|u\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \\ &= C 2^{j\frac{n-3}{2}} \|u\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}. \end{aligned} \tag{1.4.18}$$

At P, we seek to prove

$$\|K_{2,j}(x) * u(x)\|_{L^{\frac{2n^2}{n^2-3n+2}}(\mathbb{R}^n)} \leqslant C 2^{-\frac{1}{n}j} \|u\|_{L^{\frac{2n^2}{n^2+n+2}}(\mathbb{R}^n)}. \tag{1.4.19}$$

Changing the scale, replacing x with $2^j x$, we would be done if we could show

$$\|\tilde{K}_{2,j}(x) * u(x)\|_{L^{\frac{2n^2}{n^2-3n+2}}(\mathbb{R}^n)} \leqslant C 2^{-\frac{n^2-3n+2}{2n}j} \|u\|_{L^{\frac{2n^2}{n^2+n+2}}(\mathbb{R}^n)}, \tag{1.4.20}$$

where

$$\tilde{K}_{2,j}(x) = \delta(x)|x|^{-\frac{n-1}{2}} e^{-i2^j|x|\sin(\frac{1}{2}\arg z)} b(2^j x),$$

$$b(x) = e^{i\frac{n-3}{4}\arg z} e^{-|x|\cos\frac{1}{2}\arg z} a_{\frac{n-2}{2}}(|x|e^{i\frac{1}{2}\arg z}),$$

and $\delta(x)$ is as mentioned in the dyadic decomposition. The kernel $\tilde{K}_{2,j}$ is easily seen to fall within the hypotheses of the above oscillatory integral theorem, remembering that $a_{\frac{n-2}{2}}(w)$ satisfies the decaying property (1.4.9); hence applying it, we obtain the inequality (1.4.19) we were seeking, with the constant C independent of j.

The estimates at P and Q for the operator T_j enables us to utilize Bourgain's interpolation, resulting in the desired restricted weak type estimate for the operator T, which is the sum of the T_j. Specifically, what remains to do is only the following elementary computation.

$$\theta = \frac{\frac{1}{n}}{\frac{1}{n} + \frac{n-3}{2}} = \frac{2}{n^2 - 3n + 2};$$

$$\Big(\frac{n^2+n+2}{2n^2}, \frac{n^2-3n+2}{2n^2}\Big) \cdot (1-\theta) + \Big(\frac{2}{n}, 0\Big) \cdot \theta = \Big(\frac{n+1}{2n}, \frac{n-3}{2n}\Big).$$

Since the last pair of exponents is exactly the one we are aiming at, Bourgain's interpolation yields our target estimate:

$$\|K_2(x) * u(x)\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)} \leqslant C\|u\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}, \tag{1.4.21}$$

where, independent from $z \in \mathbb{C}$, the constant C depends exclusively on the dimension n. (1.4.21) together with (1.4.11) and (1.4.14) gives the conclusion of the main theorem whenever the dimension $n > 3$.

If the dimension $n = 3$, as already observed, interpolating between P and Q does not help, since Q coincides with our target pair of exponents $(\frac{2}{3}, 0)$. To remedy this, we interpolate instead between the two points $(0, 0)$ and $(1, 0)$ on the $\frac{1}{r}$ axis, bearing in mind that the oscillatory integral theorem Lemma 2 holds for the latter pair too. Therefore at $(0, 0)$, Young's inequality gives

$$\begin{aligned} \|K_{2,j}(x) * u(x)\|_{L^\infty(\mathbb{R}^n)} &\leqslant \|K_{2,j}\|_{L^1(\mathbb{R}^n)}\|u\|_{L^\infty(\mathbb{R}^n)} \\ &\leqslant C2^{-j} \cdot 2^{3j}\|u\|_{L^\infty(\mathbb{R}^n)} \\ &= C2^{2j}\|u\|_{L^\infty(\mathbb{R}^n)}, \end{aligned} \tag{1.4.22}$$

while also similar to the procedure for the case $n > 3$, the oscillatory integral theorem Lemma 2 along with a change of scale argument shows immediately that at $(1, 0)$,

$$\|K_{2,j}(x) * u(x)\|_{L^\infty(\mathbb{R}^n)} \leqslant C2^{-j}\|u\|_{L^1(\mathbb{R}^n)}. \tag{1.4.23}$$

The corresponding interpolation process is now like this:

$$\theta = \frac{1}{1+2} = \frac{1}{3},$$

$$(0,0)\cdot\theta + (1,0)\cdot(1-\theta) = (\frac{2}{3}, 0).$$

Again, the last pair of exponents is our target pair.

Recall that at the beginning of our proof, we imposed the assumption that $\mathrm{Im}z \neq 0$ and stated that the case $\mathrm{Im}z = 0$ results from continuity. Indeed, if $z = a$ is real, then by the conclusion we already have, there is a constant C, not relying on $b > 0$, such that

$$\|u\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)} \leqslant C\|(\Delta + a + ib)u\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}. \tag{1.4.24}$$

However, by Minkowski's inequality for the space $L^{\frac{2n}{n+1},1}(\mathbb{R}^n)$, noting that it is a Banach space and indeed has Minkowski's inequality, it follows that the latter term above is majorized by $C\Big[\|(\Delta + a)u\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)} + b\|u\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}\Big]$. Letting b go to 0 yields the case $\mathrm{Im}z = 0$. That concludes our proof of the main theorem. $\square$

1.5 Boundary Case of Stein-Tomas Restriction Theorem

The original Stein-Tomas restriction theorem [22], [25] (in TT^* fashion) states that if the dimension $n \geqslant 3$, one has the inequality

$$\Big\| \int_{S^{n-1}} \hat{f}(\xi) \mathrm{e}^{2\pi i\langle x,\xi\rangle}\, \mathrm{d}\sigma(\xi) \Big\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}, \tag{1.5.1}$$

at the point $(1/p, 1/q) = (\frac{n+3}{2n+2}, \frac{n-1}{2n+2})$, which is the midpoint of AB in Figure 1.3. Sogge [18] extended this result in showing that the same inequality holds for pairs of exponents (p, q) off the line of duality satisfying $1 \leqslant p < \frac{2n}{n+1}, q = \frac{n+1}{n-1}p'$. They constitute the half open line segment connecting $F(1,0)$ and $A(\frac{n+1}{2n}, \frac{(n-1)^2}{2n(n+1)})$ (exclusive). Therefore by duality and interpolation, the Stein-Tomas restriction inequality is true for pairs of exponents in the interior of the pentagon in Figure 1.3. Furthermore, Bak-Seeger [1, Proposition 2.1]) established restricted weak type inequalities at the vertices A and B

$$\Big\| \int_{S^{n-1}} \hat{f}(\xi) \mathrm{e}^{2\pi i\langle x,\xi\rangle} \mathrm{d}\sigma(\xi) \Big\|_{L^{q,\infty}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p,1}(\mathbb{R}^n)}. \tag{1.5.2}$$

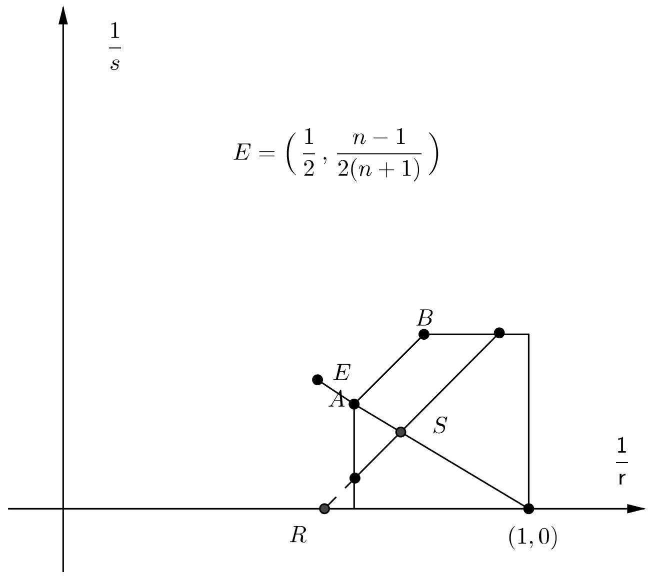

Figure 1.3: The interpolation diagram for the restriction estimates

Then by real interpolation, strong type inequalities as (1.5.1) hold on the open segment AB. In addition, strong type inequalities trivially hold on the half open segments CF and DF (excluding C and D) by Young's inequality.

However, no results seem to have been established on AC and BD before. It is clear that strong Stein-Tomas can not hold on these two segments. Indeed, in [26], radial functions belonging to $L^{\frac{2n}{n+1}}(\mathbb{R}^n)$ are constructed that have infinite Fourier transforms on S^{n-1}. Moreover, neither strong type nor restricted weak type inequality holds outside of the pentagon in Figure 1.3. In fact, if there were a restricted weak type inequality somewhere outside this pentagon, then by real interpolation, we would either get a strong inequality on the line of duality $q = p'$ with $p > \frac{2n+2}{n+3}$, or get a strong inequality somewhere on AC or BD. This is a contradiction, remembering that the range $1 \leqslant p \leqslant \frac{2n+2}{n+3}$ is sharp for a strong restriction estimate on the line of duality (see [23, p.387, 2.1.1]).

In this section, we show that restricted weak type inequality as (1.5.2) holds on the closed segments AC, BD, by an argument similar to the proof of Theorem 1. With this result and the discussion above, we completely characterize the range of (p, q) for which either strong Stein-Tomas or restricted weak type Stein-Tomas holds.

Theorem 4. *Let $n \geqslant 3$. If $p = \frac{2n}{n+1}$ and $\frac{2n(n+1)}{(n-1)^2} \leqslant q \leqslant \infty$, or $1 \leqslant p \leqslant \frac{2n(n+1)}{n^2+4n-1}$ and $q = \frac{2n}{n-1}$, then*

$$\left\| \int_{S^{n-1}} \hat{f}(\xi) \mathrm{e}^{2\pi i \langle x, \xi \rangle} \, \mathrm{d}\sigma(\xi) \right\|_{L^{q,\infty}(\mathbb{R}^n)} \leqslant C \|f\|_{L^{p,1}(\mathbb{R}^n)}$$

Proof. Our result follows from an analysis of the convolution operator whose kernel is the Fourier transform of the Lebesgue measure on the unit sphere, like in the classical case. This kernel is well-known to have the expression

$$K(x) = 2\pi|x|^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(2\pi|x|),$$

where $J_\nu(w)$ is the Bessel function, see for instance, [23, p.347-348]. Later, we will need the following fact about $J_\nu(w)$ for $\nu = m$ positive, integral or half integral, and $w = r$ real, positive and greater than 1, which is also well-known: it takes the form

$$J_m(r) = \sum_{\pm} r^{-\frac{1}{2}} e^{\pm ir} a_\pm(r), \tag{1.5.3}$$

where the functions $a_\pm(r)$, $r > 1$ are smooth and satisfy the decay property

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}r^k} a_\pm(r)\right| \leqslant C_k r^{-k}.$$

This expression can be found in [23, p.338]; see also [19, Theorem 1.2.1]. Again, dyadically decompose $K(x)$, letting $\beta_j(x)$, $j \geqslant 0$ be as in the decomposition before and $K_j(x) = \beta_j(x)K(x)$.

We first treat the case of the vertex at C, because it is exceptional. For this, we wish to apply Bourgain's interpolation to the point $O(0,0)$ and the point $F(1,0)$. There is no need to worry about the part $K_0(x)$ of $K(x)$ near the origin, since $K(x)$ is the Fourier transform of a compactly supported distribution and is thus smooth. Away from the origin, i.e., for $j > 0$, at O, Young's inequality gives

$$\|K_j(x) * f(x)\|_{L^\infty(\mathbb{R}^n)} \leqslant C 2^{\frac{n+1}{2}j} \|f\|_{L^\infty(\mathbb{R}^n)},$$

while at F, still applying Young's inequality

$$\|K_j(x) * f(x)\|_{L^\infty(\mathbb{R}^n)} \leqslant C 2^{-\frac{n-1}{2}j} \|f\|_{L^1(\mathbb{R}^n)}.$$

With the following interpolation computation,

$$\theta = \frac{\frac{n-1}{2}}{\frac{n-1}{2} + \frac{n+1}{2}} = \frac{n-1}{2n},$$

$$(1,0) \cdot (1-\theta) + (0,0) \cdot \theta = (\tfrac{n+1}{2n}, 0),$$

we obtain the restricted weak type Stein-Tomas inequality at $C(\frac{n+1}{2n}, 0)$:

$$\Big\| \int_{S^{n-1}} \hat{f}(\xi) \mathrm{e}^{2\pi i \langle x, \xi \rangle} \mathrm{d}\sigma(\xi) \Big\|_{L^\infty(\mathbb{R}^n)} \leqslant C \|f\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}.$$

Duality then produces the same restricted weak type inequality for the pair of exponents $D(1, \frac{n-1}{2n})$. However here, we cannot apply real interpolation to the points A and C, nor can we apply it to B and D, since they are along a vertical or horizontal line, which violates a hypothesis of the real interpolation theorem.

Nevertheless, we can proceed as in the proof of Theorem 1 to obtain a restricted weak type Stein-Tomas inequality at every point on the line segment joining A and C and its dual line segment joining B and D. Indeed, for each $\frac{2}{n+1} < k < \frac{n+1}{2n}$, we interpolate between the point $R(k, 0)$ and the point $S = (\frac{n-1}{2n} + \frac{n+1}{2n}k, \frac{n-1}{2n}(1-k))$, which is the intersection of the line $\frac{1}{p} - \frac{1}{q} = k$ and the line EF. At R for each $j > 0$,

$$\|K_j(x) * f(x)\|_{L^\infty(\mathbb{R}^n)} \leqslant C 2^{j(n(1-k) - \frac{n-1}{2})} \|f\|_{L^{\frac{1}{k}}(\mathbb{R}^n)},$$

while at S for $j > 0$, the familiar change of scale argument in which we replace x with $2^j x$, together with Lemma 2 produces

$$\|K_j(x) * f(x)\|_{L^s(\mathbb{R}^n)} \leqslant C 2^{j(-\frac{n+1}{2}k + 1)} \|f\|_{L^r(\mathbb{R}^n)},$$

where r and s denote the exponents corresponding to S. Finally we verify

$$\theta = \frac{\frac{n+1}{2}k - 1}{\frac{n-1}{2}(1-k)},$$

$$\Big(\frac{n-1}{2n} + \frac{n+1}{2n}k, \frac{n-1}{2n}(1-k)\Big)(1-\theta) + (k, 0)\theta = \Big(\frac{n+1}{2n}, 1 - k - \frac{n-1}{2n}\Big),$$

which gives us a restricted weak type inequality (1.5.2) at every point on the line segment AC, as we hoped. Duality then produces the same results for the dual line segment BD. $\square$

To conclude, we may summarize the results we have concerning Stein-Tomas inequality on the boundary of the pentagon. Strong Stein-Tomas holds for every pair of exponents lying on the open line segment connecting A and B, for the half open line segment connecting C and $(1, 0)$, and for the half open line segment connecting D and $(1, 0)$. Restricted weak type Stein-Tomas holds at every point on the closed line segment joining A and C, and its dual joing B and D, and counter-examples rule out the possibly of a strong Stein-Tomas at any of the points on AC and BD. Finally, no such inequalities, strong or restricted weak type, can be established outside the pentagon in Figure 1.3.

1.6 Proof of Theorem 2

As we noted in the introduction, the procedure is essentially the same as in [13], with the adaptations of the following classical results to Lorentz spaces.

1. Littlewood-Paley Theory For $t \in \mathbb{R}$, let $\chi(t)$ be the characteristic function of the set $\{t : |t| \in [1,2]\}$, and let $\chi_k(\xi_n) = \chi(2^k \xi_n)$. Suppose g is any function in $\mathscr{S}(\mathbb{R}^n)$ for simplicity. We want to show

Proposition 2. *For any $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, there exists a constant C_1, depending only on p, q and n, such that the inequality below holds*

$$\Big\| \Big(\sum_{k=-\infty}^{\infty} |\{\chi_k(\xi_n)\hat{g}(\xi)\}^{\vee}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p,q}(\mathbb{R}^n)} \leqslant C_1 \|g\|_{L^{p,q}(\mathbb{R}^n)}.$$

On the other hand, for any $1 < p < \infty$ and $1 < q \leqslant \infty$, there exists a constant C_2, depending only on p, q and n, such that the inequality below holds

$$C_2 \|g\|_{L^{p,q}(\mathbb{R}^n)} \leqslant \Big\| \Big(\sum_{k=-\infty}^{\infty} |\{\chi_k(\xi_n)\hat{g}(\xi)\}^{\vee}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p,q}(\mathbb{R}^n)}.$$

Proof. To prove this proposition, recall that Littlewood-Paley theory (see [21, Pg. 103-108]) says that

$$\Big\| \Big(\sum_{k=-\infty}^{\infty} |\{\chi_k(\xi_n)\hat{g}(\xi)\}^{\vee}|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{p}(\mathbb{R}^n)} \leqslant C_2 \|g\|_{L^{p}(\mathbb{R}^n)}$$

holds for any $1 < p < \infty$. Then real interpolation yields the same inequality for $L^{p,q}(\mathbb{R}^n)$ spaces, with $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. This is the first part of the proposition. For the second part, we imitate the proof in [21, Pg. 105], applying duality property.

Designate the set of simple functions, which is dense in $L^{p,q}(\mathbb{R}^n)$ $(1 < p < \infty, 1 \leqslant q < \infty)$, as $\mathscr{D}$. For any $h \in \mathscr{D}$, consider the identity

$$\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} \{\chi_k(\xi_n)\hat{g}(\xi)\}^{\vee} \overline{\{\chi_k(\xi_n)\hat{h}(\xi)\}^{\vee}} \mathrm{d}x = \int_{\mathbb{R}^n} g(x)\bar{h}(x)\mathrm{d}x,$$

which is a consequence of the L^2 equality

$$\sum_{k=-\infty}^{\infty} \|\{\chi_k(\xi_n)\hat{g}(\xi)\}^{\vee}\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)}.$$

An application of Schwarz's inequality and Holder's inequality, in that order, gives

$$\begin{aligned}\int_{\mathbb{R}^n} g\overline{h}\mathrm{d}x &\leqslant \int_{\mathbb{R}^n} \Big(\sum_{k=-\infty}^{\infty} |\{\chi_k(\xi_n)\hat{g}(\xi)\}^{\vee}|^2\Big)^{\frac{1}{2}}\Big(\sum_{k=-\infty}^{\infty} |\{\chi_k(\xi_n)\hat{h}(\xi)\}^{\vee}|^2\Big)^{\frac{1}{2}}\mathrm{d}x \\ &\leqslant \Big\|\Big(\sum_{k=-\infty}^{\infty} |\{\chi_k(\xi_n)\hat{g}(\xi)\}^{\vee}|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p,q}(\mathbb{R}^n)}\Big\|\Big(\sum_{k=-\infty}^{\infty} |\{\chi_k(\xi_n)\hat{h}(\xi)\}^{\vee}|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p',q'}(\mathbb{R}^n)}, \quad (1.6.1)\end{aligned}$$

where p' and q' are conjugate to p and q. Taking supremum over all $h \in \mathscr{D}$ such that $\|h\|_{L^{p',q'}(\mathbb{R}^n)} \leqslant 1$, remembering that $\mathscr{D}$ is dense in $L^{p',q'}(\mathbb{R}^n)$, we obtain $\|g\|_{L^{p,q}(\mathbb{R}^n)}$ on the left side of the above inequality. By the part of the proposition we already proved, the right side is majorized by

$$C_{p,q}\Big\|\Big(\sum_{k=-\infty}^{\infty} |\{\chi_k(\xi_n)\hat{g}(\xi)\}^{\vee}|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p,q}(\mathbb{R}^n)}.$$

This finishes the proof of the other direction. □

The reason that we didn't include $q = 1$ in the second part of the proposition is because of a lack of a good dense subspace of $L^{p',\infty}(\mathbb{R}^n)$, as we saw in Section 1.2. Nevertheless, the above proposition already suffices for our purpose.

2. Minkowski's Inequality We need the following

Proposition 3.

$$\Big\|\Big(\sum_{k=-\infty}^{\infty} |F_k(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{s,\infty}(\mathbb{R}^n)} \leqslant C\Big(\sum_{k=-\infty}^{\infty} \|F_k(x)\|^2_{L^{s,\infty}(\mathbb{R}^n)}\Big)^{\frac{1}{2}}, \quad (1.6.2)$$

for any $s > 2$, where the C depends only on s and the dimension n;

$$\Big(\sum_{k=-\infty}^{\infty} \|F_k(x)\|^2_{L^{r,1}(\mathbb{R}^n)}\Big)^{\frac{1}{2}} \leqslant C\Big\|\Big(\sum_{k=-\infty}^{\infty} |F_k(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{r,1}(\mathbb{R}^n)}, \quad (1.6.3)$$

for any $1 < r < 2$, where the C depends only on r and n.

Proof. With the preparation in Section 1.2, we have for $s > 2$,

$$\begin{aligned}
&\Big\|\Big(\sum_{k=-\infty}^{\infty}|F_k(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{s,\infty}(\mathbb{R}^n)} \leqslant C\Big\|\Big(\sum_{k=-\infty}^{\infty}|F_k(x)|^2\Big)^{\frac{1}{2}}\Big\|^{*}_{L^{s,\infty}(\mathbb{R}^n)}\\
&= C\Big(\sup_{t>0} t^s\mu\Big(\Big\{x : \Big(\sum_{k=-\infty}^{\infty}|F_k(x)|^2\Big)^{\frac{1}{2}} > t\Big\}\Big)\Big)^{\frac{1}{s}}\\
&= C\Big(\sup_{t>0} t^{\frac{s}{2}}\mu\Big(\Big\{x : \sum_{k=-\infty}^{\infty}|F_k(x)|^2 > t\Big\}\Big)\Big)^{\frac{2}{s}\cdot\frac{1}{2}}\\
&= C\Big\|\sum_{k=-\infty}^{\infty}|F_k(x)|^2\Big\|^{*\frac{1}{2}}_{L^{\frac{s}{2},\infty}(\mathbb{R}^n)}\\
&\leqslant C\Big\|\sum_{k=-\infty}^{\infty}|F_k(x)|^2\Big\|^{\frac{1}{2}}_{L^{\frac{s}{2},\infty}(\mathbb{R}^n)}\\
&\leqslant C\Big(\sum_{k=-\infty}^{\infty}\||F_k(x)|^2\|_{L^{\frac{s}{2},\infty}(\mathbb{R}^n)}\Big)^{\frac{1}{2}} \qquad (1.6.4)\\
&\leqslant C\Big(\sum_{k=-\infty}^{\infty}\||F_k(x)|^2\|^{*}_{L^{\frac{s}{2},\infty}(\mathbb{R}^n)}\Big)^{\frac{1}{2}}\\
&= C\Big(\sum_{k=-\infty}^{\infty}\Big(\sup_{t>0} t^{\frac{s}{2}}\mu(\{x : |F_k(x)|^2 > t\})\Big)^{\frac{2}{s}}\Big)^{\frac{1}{2}}\\
&= C\Big(\sum_{k=-\infty}^{\infty}\Big(\sup_{t>0} t^{s}\mu(\{x : |F_k(x)| > t\})\Big)^{\frac{2}{s}}\Big)^{\frac{1}{2}}\\
&= C\Big(\sum_{k=-\infty}^{\infty}\|F_k(x)\|^{*2}_{L^{s,\infty}(\mathbb{R}^n)}\Big)^{\frac{1}{2}}\\
&\leqslant C\Big(\sum_{k=-\infty}^{\infty}\|F_k(x)\|^{2}_{L^{s,\infty}(\mathbb{R}^n)}\Big)^{\frac{1}{2}}
\end{aligned}$$

Besides switching back and forth between the two equivalent definitions of Lorentz spaces, we performed the following operations in order in the above string of inequalities: changing variable from t to $\sqrt{t}$, applying Minkowski's inequality for the norm $\|\cdot\|_{L^{\frac{s}{2}}(\mathbb{R}^n)}$ (noting that $\frac{s}{2} > 1$), changing variable from t to t^2.

For the other part of the proposition, we make use of duality property again. Observe first that $\Big(\sum_{k=-\infty}^{k=\infty}\|F_k(x)\|^2_{L^{r,1}(\mathbb{R}^n)}\Big)^{\frac{1}{2}}$ is nothing but the l^2 norm of the sequence $\{\|F_k(x)\|_{L^{r,1}(\mathbb{R}^n)}\}_{k=-\infty}^{\infty}$, hence is equal to

$$\sup_{\sum_{k=-\infty}^{\infty}|b_k|^2\leqslant 1}\sum_{k=-\infty}^{\infty}\|F_k(x)\|_{L^{r,1}(\mathbb{R}^n)}|b_k|.$$

Then recall that for $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ and p', q' conjugate to p, q, we also have the relation

$$\sup_{\|h\|_{L^{p',q'}(\mathbb{R}^n)}\leqslant 1}\int_{\mathbb{R}^n}|g(x)h(x)|\mathrm{d}x = \|g\|_{L^{p,q}(\mathbb{R}^n)},$$

similar to what we have for L^p spaces (see (1.2.8) in Section 1.2). Taking these into account, we see that $\Big(\sum_{k=-\infty}^{\infty}\|F_k(x)\|^2_{L^{r,1}(\mathbb{R}^n)}\Big)^{\frac{1}{2}}$ is equal to

$$\sup_{\sum_{k=-\infty}^{\infty}\|G_k(x)\|^2_{L^{r',\infty}(\mathbb{R}^n)}\leqslant 1}\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^n}|F_k(x)G_k(x)|\mathrm{d}x.$$

By Schwarz's inequality and Holder's inequality, in that order, this latter expression is

$$\begin{aligned}&\leqslant\int_{\mathbb{R}^n}\Big(\sum_{k=-\infty}^{\infty}|F_k(x)|^2\Big)^{\frac{1}{2}}\Big(\sum_{k=-\infty}^{\infty}|G_k(x)|^2\Big)^{\frac{1}{2}}\mathrm{d}x\\&\leqslant\Big\|\Big(\sum_{k=-\infty}^{\infty}|F_k(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{r,1}(\mathbb{R}^n)}\Big\|\Big(\sum_{k=-\infty}^{\infty}|G_k(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{r',\infty}(\mathbb{R}^n)}.\end{aligned}$$

The last expression is majorized by $C\Big\|\Big(\sum_{k=-\infty}^{\infty}|F_k(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{r,1}(\mathbb{R}^n)}$, by the first part of the proposition, remembering our assumption on the sequence $\{\|G_k(x)\|_{L^{r',\infty}(\mathbb{R}^n)}\}$ and the fact that $r'>2$. This concludes the proof of the second part of the proposition. □

3. Stein-Tomas Inequality Recall that at the end of Section 1.5, we obtained restricted weak type Stein-Tomas inequality for the exponents (and its dual of course)

$$(\frac{1}{r},\frac{1}{s})=(\frac{n+1}{2n},\frac{n-3}{2n}),$$

which is on the line segment AC in Figure 1.3. This plays a central role in the proof of Theorem 2.

Proposition 4. *If $n\geqslant 3$, then*

$$\Big\|\int_{S^{n-1}}\hat{f}(\xi)\mathrm{e}^{2\pi i\langle x,\xi\rangle}\,\mathrm{d}\sigma(\xi)\Big\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)}\leqslant C\|f\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}$$

4. Hormander's Multipliers Theorem

Proposition 5. *Suppose that $m\in L^\infty(\mathbb{R}^n)$ satisfies, for some integer $s>\frac{n}{2}$,*

$$\sum_{0\leqslant|\alpha|\leqslant s}\sup_{\lambda>0}\lambda^{-n}\|\lambda^{|\alpha|}D^\alpha\beta(\cdot/\lambda)m(\cdot)\|^2_{L^2(\mathbb{R}^n)}<\infty,$$

whenever $\beta\in C_0^\infty(\mathbb{R}^n\backslash\{0\})$. Then for $1<p<\infty$ and $1\leqslant q\leqslant\infty$, the inequality holds

$$\|T_mf\|_{L^{p,q}(\mathbb{R}^n)}\leqslant C_{p,q}\|f\|_{L^{p,q}(\mathbb{R}^n)},$$

where the T_m is the multiplier operator with multiplier $m(x)$:

$$T_m f = \{m(\xi)\hat{f}(\xi)\}^{\check{}}.$$

Proof. Like the proof of the extension of Littlewood-Paley theory, to prove this proposition, simply recall Hormander's multipliers theorem ([19], Pg. 15), which asserts that

$$\|T_m f\|_{L^p(\mathbb{R}^n)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^n)},$$

is true for any $1 < p < \infty$. Then real interpolation gives the same inequality for Lorentz spaces. $\square$

Having the above four propositions at hand, one then is able to prove Theorem 2 easily, the proof going in the same way as [13, P.335-337].

By a reduction process similar to that at the beginning of the proof of Theorem 1 (see [13, P.335]), it suffices to prove for Theorem 2 the following special case

$$\|u\|_{L^{\frac{2n}{n=3},\infty}(\mathbb{R}^n)} \leqslant C \left\| \left\{ \Delta + 1 + \epsilon \left(\frac{\partial}{\partial x_n} + i\beta \right) \right\} u \right\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}, \tag{1.6.5}$$

where $\epsilon \neq 0$, $\beta \neq 0$. Theorem 2 is then a consequence of the estimate for a multiplier operator below:

$$\left\| \left\{ \frac{\hat{f}(\xi)}{-|\xi|^2 + 1 + i\epsilon(\xi_n + \beta)} \right\}^{\check{}} \right\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)} \leqslant C \|f\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}. \tag{1.6.6}$$

Denote the multiplier in (1.6.6) by $m(\xi)$. Also, for $t \in \mathbb{R}$, let $\chi(t)$ be the characteristic function of the set $\{t : |t| \in [1,2]\}$, and set $\chi_k(\xi_n) = \chi(2^k(\xi_n + \beta))$. For convenience, denote $\chi_k(\xi_n)m(\xi)$ as $m_k(\xi)$. It then suffices to prove a similar estimate for the multiplier $m_k(\xi)$:

$$\|\{m_k(\xi)\hat{f}(\xi)\}^{\check{}}\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)} \leqslant C \|f\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}. \tag{1.6.7}$$

Indeed, noting that $\frac{2n}{n+1} < 2 < \frac{2n}{n-3}$, if we had estimate (1.6.7), we may apply the second part of Proposition 2, the first part of Proposition 3, the second part of Proposition 3, and the first part of

Proposition 2, in that order, to obtain

$$\begin{aligned}\|\{m(\xi)\hat{f}(\xi)\}\check{}\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)} &\leqslant C\Big\|\Big(\sum_{k=-\infty}^{\infty}|\{m_k(\xi)\hat{f}(\xi)\}\check{}|^2\Big)^{\frac{1}{2}}\Big\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)}\\ &\leqslant C\Big(\sum_{k=-\infty}^{\infty}\|\{m_k(\xi)\hat{f}(\xi)\}\check{}\|^2_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)}\Big)^{\frac{1}{2}}\\ &\leqslant C\Big(\sum_{k=-\infty}^{\infty}\|\{\chi_k(\xi_n)\hat{f}(\xi)\}\check{}\|^2_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}\Big)^{\frac{1}{2}}\\ &\leqslant C\Big\|\Big(\sum_{k=-\infty}^{\infty}|\{\chi_k(\xi_n)\hat{f}(\xi)\}\check{}|^2\Big)^{\frac{1}{2}}\Big\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}\\ &\leqslant C\|f\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)},\end{aligned}$$

which is the result we are seeking.

To prove inequality (1.6.7), we first apply the special case Theorem 1 we just proved to $z = 1 + i\epsilon 2^{-k}$ and obtain

$$\Big\|\Big\{\frac{\chi_k(\xi_n)\hat{f}(\xi)}{-|\xi|^2+1+i\epsilon 2^{-k}}\Big\}\check{}\Big\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}. \tag{1.6.8}$$

By taking difference, it then remains only to demonstrate the inequality

$$\begin{aligned}&\Big\|\Big\{\frac{\chi_k(\xi_n)[i\epsilon(\xi_n+\beta-2^{-k})]\hat{f}(\xi)}{(-|\xi|^2+1+i\epsilon(\xi_n+\beta))(-|\xi|^2+1+i\epsilon 2^{-k})}\Big\}\check{}\Big\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)}\\ &\leqslant C\|f\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}.\end{aligned} \tag{1.6.9}$$

Now if we use polar coordinates $\xi = \rho\omega$, we will get, after applying Minkowski's inequality for the Lorentz space $L^{\frac{2n}{n-3}}(\mathbb{R}^n)$ and Proposition 4, the following string of inequalities

$$\begin{aligned}&\Big\|\Big\{\frac{\chi_k(\xi_n)[i\epsilon(\xi_n+\beta-2^{-k})]\hat{f}(\xi)}{(-|\xi|^2+1+i\epsilon(\xi_n+\beta))(-|\xi|^2+1+i\epsilon 2^{-k})}\Big\}\check{}\Big\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)}\\ &\leqslant \int_0^\infty \Big\|\int_{S^{n-1}}\frac{\epsilon\hat{f}(\rho\omega)\chi_k(\xi_n)(\xi_n+\beta-2^{-k})e^{i\rho\langle\omega,x\rangle}}{(-\rho^2+1+i\epsilon(\xi_n+\beta))(-\rho^2+1+i\epsilon 2^{-k})}\mathrm{d}\omega\Big\|_{L^{\frac{2n}{n-3},\infty}(\mathbb{R}^n)}\rho^{n-1}\mathrm{d}\rho\\ &\leqslant C\int_0^\infty \rho\Big\|\Big\{\frac{\epsilon\hat{f}(\xi)\chi_k(\xi_n)(\xi_n+\beta-2^{-k})}{(-\rho^2+1+i\epsilon(\xi_n+\beta))(-\rho^2+1+i\epsilon 2^{-k})}\Big\}\check{}\Big\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}\mathrm{d}\rho.\end{aligned}$$

Finally, by Proposition 5, this last expression is majorized by

$$C\|f\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}\int_0^\infty \frac{|\epsilon 2^{-k}\rho|}{(\rho^2-1)^2+(\epsilon 2^{-k})^2}\mathrm{d}\rho,$$

which is dominated by $C\|f\|_{L^{\frac{2n}{n+1},1}(\mathbb{R}^n)}$. This concludes the proof of Theorem 2.

One thing worthy of mentioning is that in Theorem 2, we exclude the case $n = 3$, in contrast with the corresponding result in Kenig-Ruiz-Sogge [13]. This is due to the fact that the exponent $\frac{2n}{n-3}$ is ∞ then, and Littlewood-Paley theory does not hold true. $\square$

As a final remark for the first part of the thesis, we point out that we have concerned only elliptic second order differential operators with constant coefficients. When the paper [16] for the above stuff was under preparation, we noticed that uniform Sobolev inequalities for nonelliptic second order differential operators with constant coefficients, those whose principal part is $Q(D)$ where

$$Q(\xi) = -\xi_1^2 - ... - \xi_k^2 + \xi_{k+1}^2 + ... + \xi_n^2, \quad 2 \leqslant k < n,$$

and $D = -i(\partial/\partial x_1, ..., \partial/\partial x_n)$, were obtained in a recent paper by Jeong-Kwon-Lee [5]. In this paper, they gave the optimal range of exponent pairs for which uniform Sobolev inequalities hold, along with corresponding unique continuation theorems.

2

Resolvent Estimates for the Laplacian on the sphere

2.1 Introduction

Since Kenig-Ruiz-Sogge's classical result on resolvent estimate in the Euclidean space, there has been a lot of interest and endeavor in extending this work to manifolds. In 2011, Dos Santos Ferreira, Kenig and Salo [8] showed that on a compact manifold M, the uniform inequality

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \leqslant C\|(\Delta_g + \zeta)u\|_{L^{\frac{2n}{n+2}}(M)}$$

is valid for all $\zeta \in \mathbb{C}$ such that $\mathrm{Im}\sqrt{\zeta} \geqslant \delta$ for a fixed positive δ. The two exponents in their result, duals of each other, correspond to the midpoint of the line segment in Figure 1.1. On a manifold of course, the Laplacian becomes the Laplace-Beltrami operator. Pictorially, the permissible $\zeta \in \mathbb{C}$ constitute the region in the complex plane outside a small ball around the origin and a parabola. This naturally poses the question of whether we are able to enlarge the region for ζ, while still obtaining a uniform estimate.

In 2013, Bourgain, Shao, Sogge and Yao [3] showed that the region in Dos Santos Ferreira-Kenig-Salo [8] is optimal for Zoll manifolds-manifolds whose geodesics are all periodic with a common minimal period, hence in particular for spheres. However, for the torus and manifolds of nonpositive curvature, they expanded the region remarkably. Later, Shao and Yao [17] proved the resolvent estimate on compact manifolds for exponent pairs that do not necessarily lie on the line of duality.

More explicitly, in their theorem, the exponents r and s need only satisfy

$$\frac{1}{r}-\frac{1}{s}=\frac{2}{n},\quad \frac{2(n+1)}{n+3}\leqslant r\leqslant\frac{2(n+1)}{n-1}.$$

This leads us to the question of whether it is possible to extend the permissible exponents to a greater part of the line segment in Figure 1.1, besides the question of finding the sharp region for ζ on certain types of manifolds.

In 2014, Huang and Sogge [12] proved the resolvent estimate on spaces of constant positive and negative curvature for exponents lying on the full line segment in Figure 1.1. The region for ζ in the case of constant positive curvature is the same as that in Dos Santos Ferreira-Kenig-Salo [8] since it was shown to be sharp, as just mentioned. The region in the case of constant negative curvature is the whole complex plane when the dimension is 3, and the complex plane with a neighborhood of the origin excluded in higher dimensions.

There has also been efforts to prove the estimate for exponent pairs that are not even on the line $\frac{1}{r}-\frac{1}{s}=\frac{2}{n}$. By an easy dilation argument, as Kenig-Ruiz-Sogge [13] pointed out, the condition $\frac{1}{r}-\frac{1}{s}=\frac{2}{n}$ is necessary in the Euclidean case in order to obtain a uniform bound independent of $\zeta\in\mathbb{C}$. Therefore, one probably should not expect to attain a uniform inequality either when it comes to manifolds. However, a bound that depends on ζ would still be of great interest, especially if it is a negative power of ζ, as this bound will tend to 0 when $|\zeta|$ goes to infinity. A recent paper by Frank and Schimmer [9] contributed just to that. They provided the following estimate on compact manifolds:

$$\|u\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)}\leqslant C|\zeta|^{-\frac{1}{n+1}}\|(\Delta+\zeta)u\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n)},$$

where as in Dos Santos Ferreira-Kenig-Salo [8], $\mathrm{Im}\sqrt{\zeta}\geqslant\delta$ for a fixed δ, and C of course, is independent of ζ. The exponents, duals of each other, lie on the line $\frac{1}{r}-\frac{1}{s}=\frac{2}{n+1}$.

Frank and Schimmer's result [9] is for general compact manifolds and concerns an exponent pair on the line of duality. We in this part of the thesis, treat the spheres only, but consider more general exponent pairs, those not necessarily on the line $\frac{1}{r}-\frac{1}{s}=\frac{2}{n}$, and not necessarily on the line of duality. Our main result is the following

Theorem 5. *Let S^n be the n dimensional sphere where $n\geqslant 3$. If the exponents r and s satisfy*

$$\frac{1}{r}-\frac{1}{s}=\sigma,\quad \frac{2}{n+1}\leqslant\sigma\leqslant\frac{2}{n},$$

$$\frac{2n}{n-1+2n\sigma} < r < \frac{2n}{n+1},$$

then we have the inequality

$$\|u\|_{L^s(S^n,\mathrm{d}V_{S^n})} \leqslant C\lambda^{n\sigma-2}\Big\|\Big(\big(\Delta_{S^n} - (\frac{n-1}{2})^2\big) + \zeta\Big)u\Big\|_{L^r(S^n,\mathrm{d}V_{S^n})}, \tag{2.1.1}$$

where $\zeta = (\lambda + i\mu)^2$, *and* $\lambda \geqslant 1$, $|\mu| \geqslant 1$.

Notice that when $\frac{2}{n+1} \leqslant \sigma < \frac{2}{n}$, the power $n\sigma - 2$ on λ in the estimate is negative. On the $(\frac{1}{r}, \frac{1}{s})$ plane, the exponent pairs in the theorem constitute the line segments connecting a point whose horizontal axis is $\frac{n+1}{2n}$ with its dual, the point whose vertical axis is $\frac{n-1}{2n}$; see Figure 2.1 below. We, as in Huang-Sogge [12], shift the Laplacian to $\Delta_{S^n} - (\frac{n-1}{2})^2$, because we can then take the square root of minus this shifted Laplacian, and the eigenvalues of the square root are $k + \frac{n-1}{2}$, $k = 0, 1, 2, \cdots$. That we are not able to prove the result for ζ in the optimal region

$$\mathcal{R} = \{\zeta \in \mathbb{C} : \mathrm{Re}\zeta \leqslant (\mathrm{Im}\zeta)^2\}$$

is because the case for small λ or $|\mu|$ were resolved by Sobolev Embedding Theorem in Huang-Sogge [12], but when the exponents are off the line $\frac{1}{r} - \frac{1}{s} = \frac{2}{n}$, the theorem no longer applies.

Let H_k denote the projection operator onto the space of spherical harmonics of degree k, i.e. the space of harmonic homogeneous polynomials of degree k, $k = 0, 1, 2, \cdots$. These are the eigenspaces of the square root of minus the shifted Laplacian, $\sqrt{-\Delta_{S^n} + (\frac{n-1}{2})^2}$, with eigenvalues $k + \frac{n-1}{2}$. We will need the following estimate on the norm of H_k stated below to prove Theorem 5.

Proposition 6. *Let* $n \geqslant 3$. *We have*

$$\|H_k\|_{L^r(S^n)\to L^s(S^n)} \leqslant Ck^{n\sigma-1}, \tag{2.1.2}$$

if the exponents r and s are as in Theorem 5, i.e.

$$\frac{1}{r} - \frac{1}{s} = \sigma, \quad \frac{2}{n+1} \leqslant \sigma \leqslant \frac{2}{n},$$

$$\frac{2n}{n-1+2n\sigma} < r < \frac{2n}{n+1}.$$

Finding the norm of H_k as an operator from $L^r(S^n)$ to $L^s(S^n)$ when $\frac{1}{r} - \frac{1}{s} = \sigma$ is interesting in its own right. For example, when $\sigma = 1$, i.e. $r = 1$ and $s = \infty$, then $\|H_k\|_{L^1(S^n)\to L^\infty(S^n)}$ is bounded

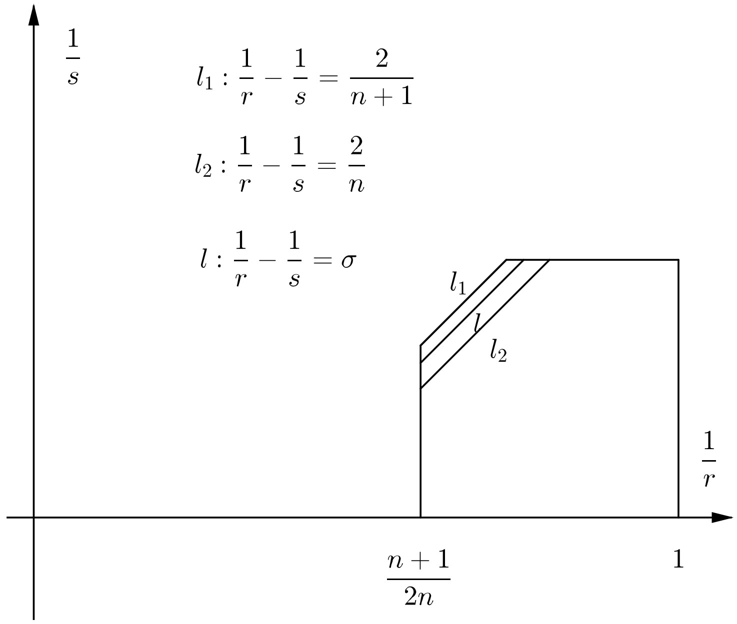

Figure 2.1: Exponents for the resolvent estimates in Theorem 5

by Ck^{n-1}, because we have the well-known estimate on the L^∞ norm of the kernel $H_k(x, y)$ of H_k (see Sogge [20], § 3.4):

$$|H_k(x, y)| \leqslant Ck^{n-1}.$$

And of course, if $r = 2$, $s = 2$, then $\|H_k\|_{L^2(S^n)\to L^2(S^n)} = 1$. Determining a bound for $\|H_k\|_{L^r(S^n)\to L^s(S^n)}$ when $\frac{1}{r} - \frac{1}{s} = \sigma$ in the form of a function of σ, e.g. k raised to a power that is a function of σ, seems then like a pretty interesting problem.

To prove the proposition, we utilize Bourgain's interpolation to prove restricted weak type inequality at the two endpoints of a fixed line segment in the proposition. Then, real interpolation yields the desired estimate on the segment between the endpoints.

2.2 Proof of Proposition 6

Fix one line segment for the exponents:

$$\frac{1}{r} - \frac{1}{s} = \sigma, \quad \frac{2n}{n-1+2n\sigma} < r < \frac{2n}{n+1}. \tag{2.2.1}$$

The essential ingredient in the proof and also in the proof of our main theorem Theorem 5 is the following lemma on an oscillatory integral, which is an extension of Proposition 2.2 in Huang-Sogge [12].

Lemma 3. *Suppose g defines a smooth Riemannian metric on $\mathbb{R}^n$ that is close to the Euclidean one and has injectivity radius larger than 10. If the function*

$$a(x,y) \in C^\infty\big(B_2(O) \times B_2(O) \backslash \{(x,x) : x \in B_2(O)\}\big)$$

satisfies the estimates

$$a(x,y) \leqslant C d_g(x,y)^{2-n}, \quad \text{if} \quad d_g(x,y) \leqslant \frac{1}{\lambda}, \tag{2.2.2}$$

$$\big|\partial^\alpha_{x,y} a(x,y)\big| \leqslant C_\alpha \lambda^{\frac{n-3}{2}} d_g(x,y)^{-\frac{n-1}{2}-|\alpha|}, \quad \text{if} \quad d_g(x,y) \geqslant \frac{1}{\lambda}, \tag{2.2.3}$$

then, for exponents as in the Proposition 6, i.e.

$$\frac{1}{r} - \frac{1}{s} = \sigma, \quad \frac{2}{n+1} \leqslant \sigma \leqslant \frac{2}{n},$$

$$\frac{2n}{n-1+2n\sigma} < r < \frac{2n}{n+1},$$

we have

$$\Big\| \int e^{i\lambda d_g(x,y)} a(x,y) f(y) \mathrm{d}y \Big\|_{L^s(B_1(O))} \leqslant C\lambda^{n\sigma-2} \|f\|_{L^r(B_1(O))}, \tag{2.2.4}$$

where $f \in C_0^\infty(\mathbb{R}^n)$ is supported in $B_1(O)$.

In the lemma, $B_2(O)$ and $B_1(O)$ are balls centered at the origin with respect to the metric g, and $d_g(x,y)$ is the Riemannian distance function induced by g.

Proof of Lemma 3 Recall that for a fixed line segment

$$\frac{1}{r} - \frac{1}{s} = \sigma, \quad \frac{2}{n+1} \leqslant \sigma \leqslant \frac{2}{n},$$

$$\frac{2n}{n-1+2n\sigma} < r < \frac{2n}{n+1},$$

we aim to prove restricted weak type inequality at the endpoints using Bourgain's interpolation, and then apply real interpolation to obtain strong estimate for exponents in between. By duality, it suffices to deal with one endpoint, say

$$(\frac{1}{p},\frac{1}{q})=(\frac{n+1}{2n},\frac{n+1-2n\sigma}{2n}).$$

In what follows, (p,q) will specifically refer to the above endpoint.

We dyadically decompose the kernel $a(x,y)$ of the oscillatory integral in the lemma. So choose a Littlewood-Paley bump function $\beta(t)\in C_0^\infty\big((\frac{1}{2},1)\big)$ such that

$$\sum_{j=-\infty}^{\infty}\beta(2^{-j}t)=1$$

whenever $t>0$. Denote the operator given by the oscillatory integral in the lemma as T, and define operators T_j, $j=0,1,2,\cdots$, by

$$T_jf(x)=\int e^{i\lambda d_g(x,y)}\beta(\lambda 2^{-j}d_g(x,y))a(x,y)f(y)\mathrm{d}y,\quad k=1,2,\cdots,$$

$$T_0=T-\sum_{j=1}^{\infty}T_j.$$

T_0 can be handled by Young's inequality, if we notice estimate (2.2.2) on $a(x,y)$ and the support property of $\beta(t)$:

$$\begin{aligned}\|T_0f(x)\|_{L^q(B_1(O))}&\leqslant C\|f\|_{L^p(B_1(O))}\Big(\int_{d_g(x,y)\leqslant\frac{1}{\lambda}}\big(d_g(x,y)^{2-n}\big)^{\frac{1}{1-\sigma}}\mathrm{d}y\Big)^{1-\sigma}\\&=C\lambda^{n\sigma-2}\|f\|_{L^p(B_1(O))}.\end{aligned}\tag{2.2.5}$$

Note that the condition $\sigma\leqslant\frac{2}{n}$ is necessary to ensure a finite bound in this estimate.

For T_k, we apply Bourgain's result Lemma 1. Recall the line segment $s=\frac{n+1}{n-1}r'$, $1\leqslant r\leqslant 2$ on which Stein's oscillatory integral theorem ([22]) holds. This line segment intersects with the one we are interested in as before,

$$\frac{1}{r}-\frac{1}{s}=\sigma,\quad\frac{2n}{n-1+2n\sigma}<r<\frac{2n}{n+1},$$

at

$$P:\Big(\frac{n+1}{2n}\sigma+\frac{n-1}{2n},-\frac{n-1}{2n}\sigma+\frac{n-1}{2n}\Big).$$

Since the oscillatory integral in Lemma 3 is closely related to the oscillatory integral in Stein's theorem, we are again naturally led to considering interpolating between the above point P of

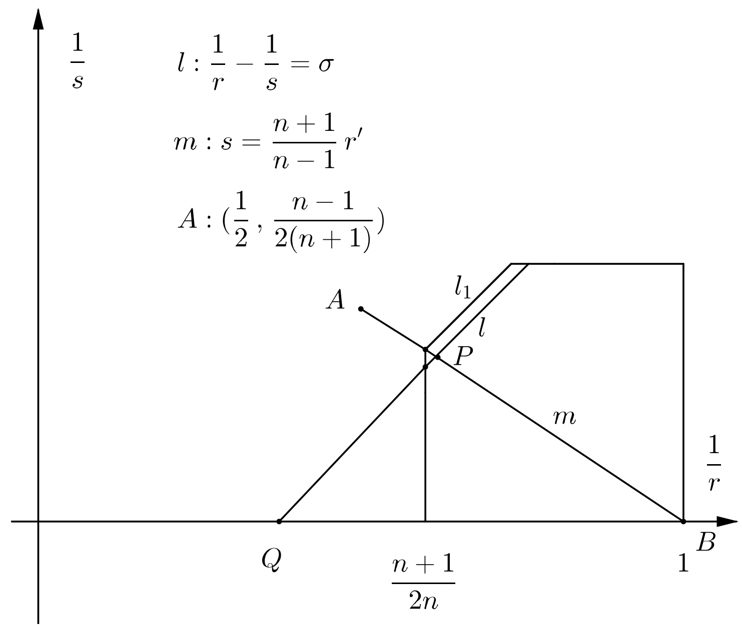

Figure 2.2: Interpolation argument 1 for the sphere

intersection with the point $Q(\sigma, 0)$, also on the line $\frac{1}{r} - \frac{1}{s} = \sigma$ but at the same time lying on the horizontal axis. These two points are shown in Figure 2.2 below. For simplicity of notation, we denote the exponents corresponding to P by (p_1, q_1) and those corresponding to Q by (p_2, q_2), so that $P = (\frac{1}{p_1}, \frac{1}{q_1})$ and $Q = (\frac{1}{p_2}, \frac{1}{q_2})$. Note that at Q, we can always obtain a trivial bound from Holder's inequality.

There is again a small obstacle that need be overcome. When $\sigma = \frac{2}{n+1}$, the point P coincides with the endpoint at which we want to prove restricted weak type estimate, hence in this case, interpolating in the above way does not help. We remedy it by interpolating instead between the two endpoints A and B of the line segment in Stein's oscillatory integral theorem. See Figure 2.2 below. In the rest of the proof, we first treat the general case, and then turn to the exceptional one.

Now we begin the interpolation process. At P, our goal is to obtain the following bound for every j:

$$\|T_j f\|_{L^{q_1}(B_1(O))} \leqslant C 2^{\left(-\frac{(n+1)\sigma}{2}+1\right) j} \lambda^{n\sigma - 2} \|f\|_{L^{p_1}(B_1(O))}, \tag{2.2.6}$$

where the bound C is independent of j. By a dilation argument, denoting $\epsilon = \frac{2^j}{\lambda}$, this would be a consequence of the estimate

$$\|S_j f\|_{L^{q_1}(B_\epsilon(O))} \leqslant C 2^{\left(\frac{n-1}{2}\sigma - \frac{n-1}{2}\right)j} \|f\|_{L^{p_1}(B_\epsilon(O))} \tag{2.2.7}$$

on the operator S_j defined by

$$S_j f(x) = \int e^{i2^j \phi_j(x,y)} b_j(x,y) f(y) \mathrm{d}y, \tag{2.2.8}$$

$$\phi_j(x,y) = \frac{1}{\epsilon} d_g(\epsilon x, \epsilon y),$$

$$b_j(x,y) = 2^{\frac{n-1}{2}j} \lambda^{2-n} \beta(\phi_j(x,y)) a(\epsilon x, \epsilon y).$$

Notice that if we introduce the new metric $g(\epsilon x)$ by dilation, then $\phi_j(x,y)$ is precisely the Riemannian distance function this new metric induces. Also, the ball of radius ϵ have radius 1 under $g(\epsilon x)$. Furthermore, the amplitude $b_j(x,y)$ vanishes if $\phi_j(x,y) \notin [\frac{1}{2}, 1]$, and satisfies the estimate

$$\left|\partial^\alpha_{x,y} b_j(x,y)\right| \leqslant C_\alpha,$$

where the bound C_α is independent of j. These observations tell that the oscillatory integral defined by the operator S_j satisfies the hypotheses of Stein's oscillatory integral theorem. Then (2.2.7), the equation we wish to prove, follows directly from Stein. We point out here that the condition $\sigma < \frac{2}{n+1}$ is necessary to ensure that the power $-\frac{(n+1)\sigma}{2} + 1$ on 2 is negative. The proof of the estimate (2.2.6) at point P has been accomplished.

As mentioned above, at Q, there is the trivial bound resulting from Holder's inequality. Noting the condition (2.2.3) on $a(x,y)$ when $d_g(x,y) > \frac{1}{\lambda}$, we readily get

$$\|T_j f\|_{L^{q_2}(B_1(O))} \leqslant C 2^{\left(\frac{n+1}{2} - n\sigma\right)j} \lambda^{n\sigma-2} \|f\|_{L^{p_2}(B_1(O))}. \tag{2.2.9}$$

When $\frac{2}{n+1} < \sigma \leqslant \frac{2}{n}$, the power $\frac{n+1}{2} - n\sigma > 0$.

It remains only some elementary computation.

$$\theta = \frac{\frac{n+1}{2} - n\sigma}{\frac{n+1}{2} - n\sigma + \frac{(n+1)\sigma}{2} - 1},$$

$$\frac{\theta}{p_1} + \frac{1-\theta}{p_2} = \frac{n+1}{2} = \frac{1}{p},$$

$$\frac{\theta}{q_1} + \frac{1-\theta}{q_2} = \frac{n+1}{2} - \sigma = \frac{1}{q}.$$

The computation shows that we have arrived exactly at $(\frac{1}{p}, \frac{1}{q})$, the endpoint that is our goal. Thus, Bourgain's interpolation gives us the desired restricted weak type inequality for the sum of T_j. Together with the strong estimate (2.2.5) for T_0, this yields

$$\|Tf\|_{L^{q,\infty}(B_1(O))} \leqslant C\lambda^{n\sigma-2}\|f\|_{L^{p,1}(B_1(O))}.$$

Duality gives the same inequality for the other endpoint. Finally, real interpolation produces the conclusion in the lemma.

There is still the exceptional case $\frac{1}{r} - \frac{1}{s} = \frac{2}{n+1}$ that need be tackled. As mentioned before, we interpolate instead between the two endpoints $A(\frac{1}{2}, \frac{n-1}{2(n+1)})$ and $B(1,0)$ in Stein's oscillatory integral theorem. The procedure is pretty much the same as with the general case, so we provide an outline. For convenience, we still denote the exponents corresponding to A and B as (p_1, q_1) and (p_2, q_2), respectively. At A, we wish to show for each j,

$$\|T_j f\|_{L^{q_1}(B_1(O))} \leqslant C2^{\frac{1}{2}j}\lambda^{-\frac{n+2}{n+1}}\|f\|_{L^{p_1}(B_1(O))}, \tag{2.2.10}$$

where the C is independent of j. By a dilation argument, this follows from

$$\|S_j f\|_{L^{q_1}(B_1(O))} \leqslant C2^{-\frac{n^2-n}{n+1}}\|f\|_{L^{p_1}(B_1(O))}, \tag{2.2.11}$$

where S_j is as in (2.2.8), and the unit ball $B_1(O)$ is with respect to the dilated metric $g(\epsilon x)$. But (2.2.11) is a direct consequence of Stein's oscillatory integral theorem, so our work at point A is done. At B, we have the trivial estimate

$$\|T_j f\|_{L^{q_2}(B_1(O))} \leqslant C2^{-\frac{n-1}{2}j}\lambda^{n-2}\|f\|_{L^{p_2}(B_1(O))}. \tag{2.2.12}$$

Note again that the power $-\frac{n-1}{2}$ of 2 in (2.2.12) is negative. Then we verify

$$\theta = \frac{\frac{n-1}{2}}{\frac{n-1}{2} + \frac{1}{2}},$$

$$\theta(\frac{1}{p_1}, \frac{1}{q_1}) + (1-\theta)(\frac{1}{p_2}, \frac{1}{q_2}) = (\frac{n+1}{2n}, \frac{(n-1)^2}{2n(n+1)}).$$

Once more, this latter pair of exponents correspond exactly to one endpoint of the extraordinary line segment. Therefore, restricted weak type inequality holds at this endpoint

$$\|Tf\|_{L^{\frac{2n(n+1)}{(n-1)^2},\infty}(B_1(O))} \leqslant C\lambda^{-\frac{2}{n+1}} \|f\|_{L^{\frac{2n}{n+1},1}(B_1(O))}. \tag{2.2.13}$$

The rest of the work, i.e. an application of duality and real interpolation, is the same as in the general case. This finishes the entire proof of Lemma 3. □

Proof of Proposition 6 With Lemma 3, we can proceed to show Proposition 6. The procedure parallels that in Huang-Sogge [12]. By the compactness of the sphere, we may assume all functions f appearing in this proof to be supported in a ball of radius 1. It is well-known that $H_k(x,y) \leqslant Ck^{n-1}$ ([20], § 3.4). However, to prove the proposition, we need a more precise bound on $H_k(x,y)$, at least for large k. The result we cited is Proposition 2.1 in Huang-Sogge [12].

Proposition 7. *When k is large,*

$$H_k(x,y) = \lambda_k^{\frac{n-1}{2}} \sum_{\pm} a_\pm(k;x,y) e^{\pm i\lambda_k d_{S^n}(x,y)},$$
$$\text{for} \quad \frac{1}{\lambda_k} \leqslant d_{S^n}(x,y) \leqslant \frac{3\pi}{4}, \tag{2.2.14}$$

where $a_\pm(k;x,y)$ are smooth functions satisfying for each $j = 0,1,2,\cdots,$

$$\left|\partial_{x,y}^j a_\pm(k;x,y)\right| \leqslant C d_{S^n}(x,y)^{-j}, \quad \text{if} \quad \frac{1}{\lambda_k} \leqslant d_{S^n}(x,y) \leqslant \frac{3\pi}{4}. \tag{2.2.15}$$

We also have the expression

$$H_k(x,y) = (-1)^k \lambda_k^{\frac{n-1}{2}} \sum_{\pm} a_\pm(k;x,y^*) e^{\pm i\lambda_k d_{S^n}(x,y^*)},$$
$$\text{for} \quad \frac{\pi}{4} \leqslant d_{S^n}(x,y) \leqslant \pi - \frac{1}{\lambda_k}. \tag{2.2.16}$$

Choose $\alpha(t) \in C^\infty(\mathbb{R}_+)$ such that

$$\alpha(t) = 1 \quad \text{if} \quad t \leqslant \delta; \quad \alpha(t) = 0 \quad \text{if} \quad t \geqslant 2\delta,$$

where δ is to be specified later. Define

$$\tilde{H}_k(x,y) = \alpha(d_{S^n}(x,y)) H_k(x,y).$$

If δ is sufficiently small, then by Proposition 7 and the estimate $H_k(x,y) \leqslant Ck^{n-1}$, it is easy to see that $\frac{\tilde{H}_k(x,y)}{\lambda_k}$ satisfies that hypotheses set for the amplitude $a(x,y)$ in Lemma 3. Therefore, applying that lemma yields

$$\|\tilde{H}_k f\|_{L^s(S^n)} \leqslant k^{n\sigma-1}\|f\|_{L^r(S^n)}, \tag{2.2.17}$$

where the exponents r and s are on our fixed line segment (2.2.1). For the same reason, if we define

$$\tilde{H}_k^*(x,y) = \alpha(d_{S^n}(x,y^*))H_k(x,y),$$

we have

$$\|\tilde{H}_k^* f\|_{L^s(S^n)} \leqslant k^{n\sigma-1}\|f\|_{L^r(S^n)}. \tag{2.2.18}$$

What remains is $U_k = H_k - \tilde{H}_k - \tilde{H}_k^*$, an operator given by the oscillatory integral

$$U_k f(x) = \lambda_k^{\frac{n-1}{2}} \sum_{\pm} \int_{S^n} \alpha_{\pm}(k;x,y) e^{\pm i\lambda_k d_{S^n}(x,y)} f(y)\mathrm{d}y,$$

where the functions $\alpha_{\pm}(k;x,y)$ vanishes when $d_{S^n}(x,y) \notin [\delta, \pi-\delta]$, and satisfies

$$|\partial_{x,y}^{\gamma}\alpha_{\pm}(k;x,y)| \leqslant C_{\gamma},$$

with the bound C_γ independent of k. This reminds us to utilize again Stein's oscillatory integral theorem. We apply it to the pair of exponents $\left(\frac{n+1}{2n}, \frac{(n-1)^2}{2n(n+1)}\right)$, which is point C in Figure 2.3 below. The result is

$$\|U_k f\|_{L^{\frac{2n(n+1)}{(n-1)^2}}(S^n)} \leqslant C\lambda_k^{\frac{n-1}{n+1}}\|f\|_{L^{\frac{2n}{n+1}}(S^n)}. \tag{2.2.19}$$

Furthermore, at $D(\frac{n+1}{2n}, 0)$, the point on the horizontal axis also labeled in Figure 2.3, we have an estimate from Holder's inequality

$$\|U_k f\|_{L^\infty(S^n)} \leqslant C\lambda_k^{\frac{n-1}{2}}\|f\|_{L^{\frac{2n}{n-1}}(S^n)}. \tag{2.2.20}$$

Interpolating between C and D then gives the desired strong estimate at one endpoint of the line segment (2.2.1):

$$\|U_k f\|_{L^{\frac{2n}{n+1-2n\sigma}}(S^n)} \leqslant C\lambda^{n\sigma-1}\|f\|_{L^{\frac{2n}{n+1}}(S^n)}. \tag{2.2.21}$$

As before, duality produces the inequality at the other endpoint, and another interpolation results in the estimate we are seeking for exponents between the endpoints. Combining (2.2.17), (2.2.18)

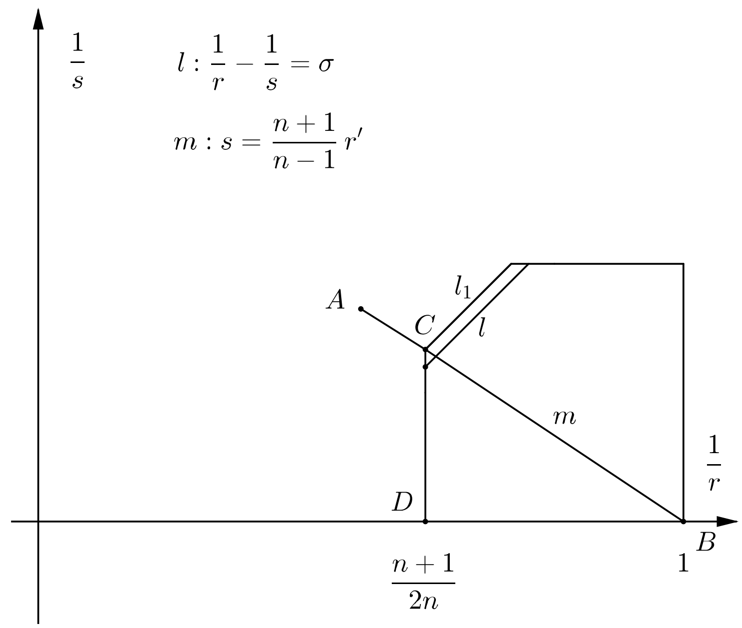

Figure 2.3: Interpolation argument 2 for the sphere

and (2.2.21), we finish our proof of Proposition 6. $\square$

2.3 Proof of Theorem 5

We finally come to the proof of the main theorem of the second part. It follows the lines in Huang-Sogge [12]. Let P represent the operator $\sqrt{-\Delta_{S^n} + (\frac{n-1}{2})^2}$. We apply the formula

$$(P^2 + (\lambda + i\mu)^2)^{-1} = \frac{\mathrm{sgn}\mu}{i(\lambda + i\mu)} \int_0^\infty e^{i(\mathrm{sgn}\mu)\lambda t} e^{-|\mu|t} \cos(tP) \mathrm{d}t, \tag{2.3.1}$$

which can be found in [3].

Choose a nonnegative even function $\rho(t) \in C_0^\infty(\mathbb{R})$ that satisfies

$$\rho(t) = 1 \quad \text{if} \quad |t| < \frac{1}{2}; \quad \rho(t) = 0 \quad \text{if} \quad |t| \geqslant 1.$$

By Proposition 2.4 in Huang-Sogge [12], the kernel of the operator

$$R_0 f = \frac{\mathrm{sgn}\mu}{i(\lambda + i\mu)} \int_0^\infty \rho(t) e^{i(\mathrm{sgn}\mu)\lambda t} e^{-|\mu|t} \cos(tP) \mathrm{d}t$$

has the expression

$$\sum_{\pm} b_{\pm}(\lambda; x, y) e^{\pm i\lambda d_{S^n}(x,y)} + O\big((d_{S^n}(x,y))^{2-n}\big),$$

where the functions $b_{\pm}(\lambda; x, y)$ vanish when $d_{S^n}(x, y)$ is near to π, and satisfy the estimates (2.2.2) and (2.2.3) in Lemma 3, with all constants independent of $\lambda \geqslant 1$. Therefore, Lemma 3 and Young's inequality give

$$\|R_0 f\|_{L^s(S^n)} \leqslant C_{r,s} \lambda^{n\sigma-2} \|f\|_{L^r(S^n)}. \tag{2.3.2}$$

For the remaining "$1 - \rho(t)$" part, denote

$$m_{\lambda,\mu}(\tau) = \frac{\mathrm{sgn}\mu}{i(\lambda + i\mu)} \int_0^\infty (1 - \rho(t)) e^{i(\mathrm{sgn}\mu)\lambda t} e^{-|\mu| t} \cos(t\tau) \mathrm{d}t.$$

Then, an easy integration by parts argument shows

$$|m_{\lambda,\mu}(\tau)| \leqslant C_N \lambda^{-1} (1 + |\lambda - \tau|)^{-N}$$

for every natural N, whenever $\tau \geqslant 0$, and $\lambda, |\mu| \geqslant 1$. Therefore, by the spectral theorem,

$$\begin{aligned}
\Big\| \big((-\Delta_{S^n} + (\frac{n-1}{2})^2)^{-1} - R_0\big) f \Big\|_{L^s(S^n)} &= \Big\| \sum_{k=0}^{\infty} m_{\lambda,\mu}(\lambda_k) H_k f \Big\|_{L^s(S^n)} \\
&\leqslant \sum_{k=0}^{\infty} \|m_{\lambda,\mu}\|_{L^\infty} \|H_k f\|_{L^s(S^n)} \\
&\leqslant C \sum_{k=0}^{\infty} \lambda^{-1} (1 + |\lambda - k|)^{-3} k^{n\sigma - 1} \|f\|_{L^r(S^n)} \\
&\leqslant C \lambda^{n\sigma-2} \|f\|_{L^r(S^n)}.
\end{aligned} \tag{2.3.3}$$

This concludes the proof of Theorem 5. $\square$

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Curriculum Vitae

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